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In this chapter, we will study the direction cosines, direction ratios of a line joining two points and discuss about the different forms of equations of lines in space under different conditions using vector algebra.

THREE-DIMENSIONAL GEOMETRY

|TOPIC 1|

Direction Cosines and Direction Ratios of a Line

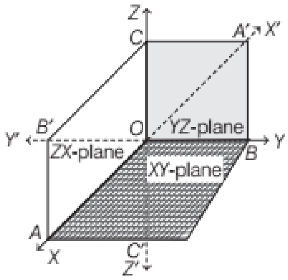
We know that the position of a point in a plane can be determined, if the coordinates (x, y) of the point with reference to two mutually perpendicular lines called X and Y-axes are known. In order to locate a point in space, two coordinate axes is insufficient. So, we need three coordinate axes called X, Y and Z-axes.

CHAPTER CHECKLIST

- Direction Cosines and Direction Ratios of a Line
- Lines in Space

Coordinate Axes and Coordinate Planes in Three-Dimensional Geometry

Let $X'OX$, $Y'OY$ and $Z'OZ$ be three mutually perpendicular lines intersecting at O. The point O is called the **origin** and the lines $X'OX$, $Y'OY$ and $Z'OZ$ are called X-axis, Y-axis and Z-axis, respectively. These three lines are also called the **rectangular coordinate axes**. These lines constitute the rectangular coordinate system.

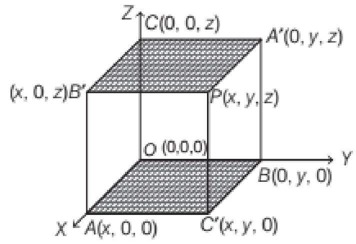


These three axes, taken in pairs determine three mutually perpendicular planes XOY, YOZ and ZOX or simply XY-plane, YZ-plane and ZX-plane, called **rectangular coordinate planes** which divide the space into eight parts called **octants**.

Note Equation of YZ-plane is $x = 0$, equation of ZX-plane is $y = 0$ and equation of XY-plane is $z = 0$.

COORDINATES OF A POINT IN SPACE

The coordinates of a point are the distances from the origin to the feet of the perpendiculars, drawn from the point on the respective coordinate axes.



The coordinates of the origin O are $(0, 0, 0)$. The coordinates of any point on the X-axis, Y-axis and Z-axis will be as $A(x, 0, 0)$, $B(0, y, 0)$ and $C(0, 0, z)$, respectively and the coordinates of any point in space will be $P(x, y, z)$.

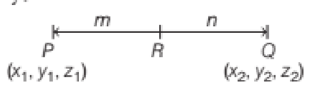
Note Fact about coordinates of point $P(x, y, z)$
x = Perpendicular distance of P from YZ-plane
y = Perpendicular distance of P from XZ-plane
z = Perpendicular distance of P from XY-plane

Important Formulae

(i) **Distance between two points** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points referred to a system of rectangular axes, then

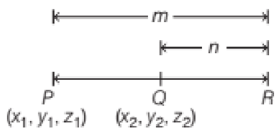
$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

(ii) Section formula Let the two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and the point $R(x, y, z)$ divides PQ in the given ratio $m:n$ internally, then



$$R = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

If the point R divides PQ externally in the ratio $m:n$, then



$$R = \left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$$

(iii) Mid-point of a line The mid-point of the line joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

(iv) Centroid of a triangle The centroid of ΔABC with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$C = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

Note **Centroid of a tetrahedron** The coordinates of the centroid of a tetrahedron, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4), are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

EXAMPLE |1| Find the distance of a point $P(a, b, c)$ from X-axis. **[All India 2014C]**

Sol. Draw a perpendicular line from P to the X-axis, then coordinates of intersection point is $Q(a, 0, 0)$.

$\therefore$ Required distance,

$$PQ = \sqrt{(a - a)^2 + (0 - b)^2 + (0 - c)^2}$$

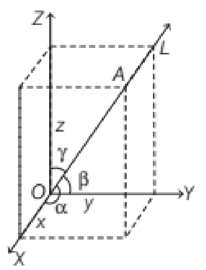
$$\left[\because \text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\right]$$

$$= \sqrt{0 + b^2 + c^2} = \sqrt{b^2 + c^2}$$

DIRECTION COSINES OF A LINE

As we know that if a directed line (say L) passing through the origin makes angles α, β and γ with X, Y and Z-axes respectively, called **direction angles**, then cosine values of these angles, i.e. $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as the **direction cosines** of the **directed line** L (or $\overrightarrow{OA}$).

The direction cosines are represented by l, m and n. Thus, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.



If the direction of directed line is reversed (i.e. opposite), then the direction angles are replaced by their supplements, i.e. $\pi-\alpha$, $\pi-\beta$ and $\pi-\gamma$. Also, the signs of direction cosines are reversed, i.e. $(-l, -m, -n)$.

Note

(i) A line in space can be extended in two opposite directions and so it has two sets of direction cosines. To have a unique set of direction cosines for a given line in space, we must take the given line as a directed line.

(ii) If the given line in space does not pass through the origin, then in order to find its direction cosines, we draw a line through the origin and parallel to the given line. Now, take one of the directed lines from the origin and find its direction cosines and then use the result that two parallel lines have same set of direction cosines.

(iii) (a) DC's of the X-axis are 1, 0, 0.
(b) DC's of the Y-axis are 0, 1, 0.
(c) DC's of the Z-axis are 0, 0, 1.

EXAMPLE |2| If a line makes angles 90°, 135°, 45° with the X, Y and Z-axes, respectively. Find its direction cosines. **[NCERT]**

Sol. Let direction cosines of the line be l, m and n.

Given, $\alpha = 90^\circ, \beta = 135^\circ$ and $\gamma = 45^\circ$

Then, $l = \cos\alpha = \cos 90^\circ = 0,$

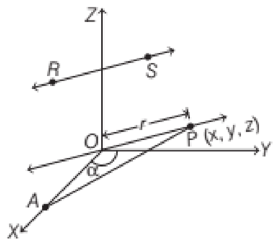
$$m = \cos\beta = \cos 135^\circ = \frac{-1}{\sqrt{2}}$$

and $$n = \cos\gamma = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

Hence, the direction cosines of a line are $0, \frac{-1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$.

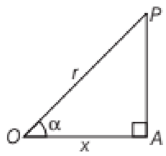
Relation between Direction Cosines of a Line

Let direction cosines of a line RS be l, m and n. Now, draw a line passing through origin and parallel to the given line. Take a point $P(x, y, z)$ on this line and draw a perpendicular PA from P on X-axis.



Let $OP = r$. Then, in right angled ΔOAP,

$$\cos\alpha = \frac{OA}{OP} = \frac{x}{r}$$



$\Rightarrow \quad x = lr$

Similarly, $y = mr$ and $z = nr$

Now, $x^2 + y^2 + z^2 = r^2(l^2 + m^2 + n^2)$

$\Rightarrow \quad r^2 = r^2(l^2 + m^2 + n^2)$

$[\because$ distance $OP = r \Rightarrow x^2 + y^2 + z^2 = r^2]$

$$\Rightarrow \quad \boxed{l^2 + m^2 + n^2 = 1}$$

which is the required relation between direction cosines of a line.

EXAMPLE |3| Find the direction cosines of a line which makes equal angles with coordinate axes. **[NCERT; All India 2019]**

Sol. Let the line makes an angle α with each of the three coordinate axes, then its direction cosines are $l = \cos\alpha$, $m = \cos\alpha$ and $n = \cos\alpha$.

We know that $l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = \frac{1}{3} \Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

Hence, direction cosines of a line are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ or $\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$.

Direction Cosines of a Line Passing through Two Points

The direction cosines of a line passing through the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are given by

$$\frac{x_2 - x_1}{AB}, \frac{y_2 - y_1}{AB} \text{ and } \frac{z_2 - z_1}{AB}$$

where, $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

DIRECTION RATIOS OF A LINE

Any three numbers a, b and c proportional to the direction cosines l, m and n respectively, are called the direction ratios or direction numbers of the line.

Suppose, $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say), k being a constant.

$\Rightarrow \quad l = ak, m = bk$ and $n = ck$

But we know that $l^2 + m^2 + n^2 = 1$

$\therefore (ak)^2 + (bk)^2 + (ck)^2 = 1 \Rightarrow k^2(a^2 + b^2 + c^2) = 1$

$$\Rightarrow \quad k = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore \quad l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \; m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

and $$n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Thus, if a, b and c are direction ratios, then direction cosines are $\left(\frac{\pm a}{\sqrt{a^2+b^2+c^2}}, \frac{\pm b}{\sqrt{a^2+b^2+c^2}}, \frac{\pm c}{\sqrt{a^2+b^2+c^2}}\right)$, where signs should be taken all positive or all negative.

For any line, if a, b and c are direction ratios of a line, then ka, kb, kc; $k \neq 0$ is also a set of direction ratios. So, any two sets of direction ratios of a line are also proportional.

Hence, for a line, there are infinitely many sets of direction ratios.

Direction Ratios of a Line Passing through Two Points

The direction ratios of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$.

Note Direction ratios of two parallel lines are proportional, as two parallel lines have same set of direction cosines.

EXAMPLE |4| If a line has direction ratios 2, −1, 2, then determine its direction cosines. **[NCERT]**

Sol. Given direction ratios are (2, −1, 2), i.e. $a = 2$, $b = -1$ and $c = 2$.

Then, $\sqrt{a^2+b^2+c^2} = \sqrt{(2)^2+(-1)^2+(2)^2} = \sqrt{9} = 3$

Now, direction cosines are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{2}{3},$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-1}{3}$$

and $$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{2}{3}$$

EXAMPLE |5| Find the direction ratios and direction cosines of the line passing through two points (2, −4, 5) and (0, 1, −1).

Sol. Let $A(x_1, y_1, z_1) = (2, -4, 5)$, $B(x_2, y_2, z_2) = (0, 1, -1)$.

Then, DR's of line AB is $(0-2, 1+4, -1-5)$, i.e. $(-2, 5, -6)$

Now, DC's of AB are

$$\left(\frac{-2}{\sqrt{(-2)^2+5^2+(-6)^2}}, \frac{5}{\sqrt{(-2)^2+5^2+(-6)^2}}, \frac{-6}{\sqrt{(-2)^2+5^2+(-6)^2}}\right)$$

Hence, DC's of line AB are $\left(\frac{-2}{\sqrt{65}}, \frac{5}{\sqrt{65}}, \frac{-6}{\sqrt{65}}\right)$.

EXAMPLE |6| Find the direction cosines of the sides of the triangle whose vertices are (3, 5, −4), (−1, 1, 2) and (−5, −5, −2). **[NCERT]**

Sol. Let the vertices of ΔABC be $A(3, 5, -4)$, $B(-1, 1, 2)$ and $C(-5, -5, -2)$.

Then, the direction ratios of side AB are $[(-1-3), (1-5), 2-(-4)]$, i.e. $(-4, -4, 6)$.

[$\because$ if the given points are $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, then DR's of $AB = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$]

and the direction cosines of AB are

$$\left(\frac{-4}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}}\right) \text{ or } \left(\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$$

[$\because$ if a, b and c are direction ratios, then direction cosines are $\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$]

Similarly, the direction ratios of side BC are $[\{-5-(-1)\}, (-5-1), (-2-2)]$, i.e. $(-4, -6, -4)$

and the direction cosines of BC are

$$\left(\frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}\right) \text{ or } \left(\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right)$$

The direction ratios of side AC are $[(-5-3), (-5-5), \{(-2-(-4)\}]$, i.e. $(-8, -10, 2)$

and the direction cosines of AC are

$$\left(\frac{-8}{2\sqrt{42}}, \frac{-10}{2\sqrt{42}}, \frac{2}{2\sqrt{42}}\right) \text{ or } \left(\frac{-4}{\sqrt{42}}, \frac{-5}{\sqrt{42}}, \frac{1}{\sqrt{42}}\right)$$

CONDITION FOR COLLINEARITY OF THREE POINTS

Suppose $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are three points in a space. Then, direction ratios of line joining A and B, B and C are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ and $(x_3 - x_2, y_3 - y_2, z_3 - z_2)$, respectively. If direction ratios of AB and BC are proportional, then these points are collinear, otherwise not.

EXAMPLE |7| Show that the points $A(2, 3, -4)$, $B(1, -2, 3)$ and $C(3, 8, -11)$ are collinear. **[NCERT]**

Sol. Given points are $A(2, 3, -4)$, $B(1, -2, 3)$ and $C(3, 8, -11)$.

Direction ratios of line joining A and B are

$$[1-2, -2-3, 3-(-4)], \text{ i.e. } (-1, -5, 7).$$

Direction ratios of line joining B and C are

$$(3-1, 8+2, -11-3), \text{ i.e. } (2, 10, -14).$$

Now, ratios of DR's of AB and BC are

$$\frac{-1}{2}, \frac{-5}{10}, \frac{7}{-14}, \text{ i.e. } -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}.$$

Thus, the direction ratios of AB and BC are proportional. So, AB is parallel to BC. But B is common to both AB and BC.

Hence, A, B and C are collinear points.

TOPIC PRACTICE 1

OBJECTIVE TYPE QUESTIONS

1 The coordinates of the foot of the perpendicular drawn from the point (2, 5, 7) on the X-axis are given by **[NCERT Exemplar]**

(a) (2, 0, 0) (b) (0, 5, 0)
(c) (0, 0, 7) (d) (0, 5, 7)

2 Distance of the point (α, β, γ) from Y-axis is **[NCERT Exemplar]**

(a) β (b) $|\beta|$
(c) $|\beta| + |\gamma|$ (d) $\sqrt{\alpha^2 + \gamma^2}$

3 If the direction cosines of a line are k, k and k, then **[NCERT Exemplar]**

(a) $k > 0$ (b) $0 < k < 1$
(c) $k = 1$ (d) $k = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

4 A line makes the same angle θ with each of the X and Z-axes. If the angle β, which it makes with Y-axis, is such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals

(a) $\frac{2}{5}$ (b) $\frac{1}{5}$ (c) $\frac{3}{5}$ (d) $\frac{2}{3}$

5 The direction ratios of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

(a) $x_1 - x_2, y_1 - y_2, z_1 - z_2$ (b) $x_2 - x_1, y_2 - y_1, z_2 - z_1$
(c) Both (a) and (b) (d) None of these

VERY SHORT ANSWER Type Questions

6 Find the distance of a point (2, 3, 4) from X-axis. **[Delhi 2010C]**

7 Write the direction cosines of a line parallel to Z-axis. **[Foreign 2012]**

8 If a line makes angles 90°, 60° and 30° with the positive direction of X, Y and Z-axes respectively, then find its direction cosines.

9 If a line makes angles 90°, 60° and θ with X, Y and Z-axes respectively, where θ is acute, then find θ. **[Delhi 2015]**

10 If a line makes angles 90° and 60° respectively with the positive directions of X and Y-axes, find the angle which it makes with the positive direction of Z-axis. **[Delhi 2017]**

11 If a line makes angles α, β and γ with the positive direction of coordinate axes, then write the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$. **[Delhi 2015C]**

12 Find the direction cosines of a line whose direction ratios are 2, −6, 3.

13 If $P = (1, 5, 4)$ and $Q = (4, 1, -2)$, then find the direction ratios of PQ.

14 Find the direction cosines of the line segment joining the points $A(7, -5, 9)$ and $B(5, -3, 8)$.

15 Find the direction ratios of a line whose direction cosines are $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$.

SHORT ANSWER Type I Questions

16 Given that $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divides PR.

17 If a line in the space makes angles α, β and γ with the coordinate axes, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

18 If a line makes angles α, β and γ with the coordinate axes, then prove that $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$.

HINTS & SOLUTIONS

1. (a) **Hint** The coordinates of the foot of perpendicular drawn from the point $P(x, y, z)$ on the X-axis are $(x, 0, 0)$.

2. (d) Required distance $= \sqrt{(\alpha - 0)^2 + (\beta - \beta)^2 + (\gamma - 0)^2}$
$= \sqrt{\alpha^2 + \gamma^2}$

3. (d) Since, direction cosines of a line are k, k and k.

$\therefore \quad l = k,\ m = k$ and $n = k$

We know that $\quad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad k^2 + k^2 + k^2 = 1 \Rightarrow k^2 = \frac{1}{3}$

$\therefore \quad k = \pm\frac{1}{\sqrt{3}}$

4. (c) Clearly, $\cos^2\theta + \cos^2\beta + \cos^2\theta = 1$

$\Rightarrow \quad 2\cos^2\theta + 1 - \sin^2\beta = 1 \Rightarrow 2\cos^2\theta - \sin^2\beta = 0$

$\Rightarrow \quad 2\cos^2\theta - 3\sin^2\theta = 0 \quad [\because \sin^2\beta = 3\sin^2\theta \text{ (given)}]$

$\Rightarrow \quad \tan^2\theta = \frac{2}{3}$

$\therefore \quad \cos^2\theta = \frac{1}{1 + \tan^2\theta} = \frac{1}{1 + \frac{2}{3}} = \frac{3}{5}$

5. (c) The direction ratios of the line segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as $x_2 - x_1, y_2 - y_1, z_2 - z_1$ or $x_1 - x_2, y_1 - y_2, z_1 - z_2$.

6. Similar as Example 1. [**Ans.** 5 units]

7. As we know that two parallel lines have same set of direction cosines. Therefore, required direction cosines are the direction cosines of Z-axis, i.e. 0, 0, 1.

8. Similar as Example 2. $\left[\textbf{Ans. } 0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right]$

9. Given, $\alpha = 90°, \beta = 60°$ and $\gamma = \theta$. Then,

$$\cos^2 90° + \cos^2 60° + \cos^2\theta = 1$$
$$[\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1]$$
$$\Rightarrow \quad 0 + \frac{1}{4} + \cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{3}{4} \Rightarrow \cos\theta = \pm\frac{\sqrt{3}}{2}$$
$$\Rightarrow \quad \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30° \qquad [\because \theta \text{ is an acute angle}]$$

10. Solve as Question 9. [**Ans.** 30°, 150°]

11. Since, the line makes angles α, β and γ with positive direction of coordinate axes, therefore $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of the line.

So, we have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
$$\Rightarrow 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$
$$[\because \sin^2 x + \cos^2 x = 1]$$
$$\Rightarrow 3 - 1 = \sin^2\alpha + \sin^2\beta + \sin^2\gamma$$
Hence, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

12. Similar as Example 4. $\left[\textbf{Ans. } \frac{2}{7}, \frac{-6}{7}, \frac{3}{7}\right]$

13. Similar as Example 5. [**Ans.** 3, −4, −6]

14. Similar as Example 5. $\left[\textbf{Ans. } -\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right]$

15. As we know that any three numbers proportional to the direction cosines are called the direction ratios.

So, infinite sets of direction ratios can be found, one of them is $1, \sqrt{2}, 1$. [multiplying direction cosines by 2]

16. Let Q divides PR in the ratio $k:1$. Then, the coordinates of Q are $\left(\frac{9k+3}{k+1}, \frac{8k+2}{k+1}, \frac{-10k-4}{k+1}\right)$.

But it is given that coordinates of Q are (5, 4, −6).
$$\therefore \quad \frac{9k+3}{k+1} = 5, \quad \frac{8k+2}{k+1} = 4, \quad \frac{-10k-4}{k+1} = -6$$
On solving all these equations, we get $k = 1/2$.

So, Q divides PR in the ratio 1 : 2.

17. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \sin^2\alpha + \sin^2\beta + \sin^2\gamma$
$$= (\cos^2\alpha - \sin^2\alpha) + (\cos^2\beta - \sin^2\beta) + (\cos^2\gamma - \sin^2\gamma)$$
$$+ \sin^2\alpha + \sin^2\beta + \sin^2\gamma \quad [\because \cos 2\theta = \cos^2\theta - \sin^2\theta]$$
$$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

18. **Hint** Use $\cos 2\theta = 2\cos^2\theta - 1$.

Now, solve as Question 17.

|TOPIC 2|

Lines in Space

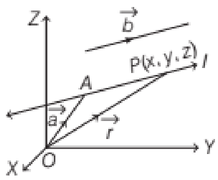
A line (or straight line) is a curve such that all the points on the line segment joining any two points of it lies on it. A line in space can be determined uniquely, if

(i) its direction and the coordinates of a point on it are known.

(ii) it passes through two given points.

Equation of a Line through a Given Point and Parallel to a Given Vector

VECTOR EQUATION



The vector equation of a line l passing through a point A with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{r}$ is the position vector of an arbitrary point P on the line and λ is a real number.

The vector equation of a straight line passing through the origin and parallel to given vector $\vec{b}$ is

$$\boxed{\vec{r} = \lambda\vec{b}}$$

Note

(i) If $\vec{b} = a'\hat{i} + b'\hat{j} + c'\hat{k}$, then a', b' and c' are direction ratios of the line and conversely, if a', b' and c' are direction ratios of a line, then $\vec{b} = a'\hat{i} + b'\hat{j} + c'\hat{k}$ will be parallel to the line.

(ii) If $\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$, then coordinates of any point on the line are given by considering $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and comparing the coefficient of $\hat{i}$, $\hat{j}$ and $\hat{k}$, i.e. first write
$$x\hat{i} + y\hat{j} + z\hat{k} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$
$$= (a_1 + \lambda b_1)\hat{i} + (a_2 + \lambda b_2)\hat{j} + (a_3 + \lambda b_3)\hat{k}$$
$$\Rightarrow \quad x = a_1 + \lambda b_1;\ y_2 = a_2 + \lambda b_2 \text{ and } z = a_3 + \lambda b_3$$
Then, $(x = a_1 + \lambda b_1, y = a_2 + \lambda b_2, z = a_3 + \lambda b_3)$ represent the coordinates of any point on the line.

CARTESIAN EQUATION

The cartesian equation of a line passing through a point $A(x_1, y_1, z_1)$ and having direction ratios a, b and c is

$$\boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}}$$

where, (x, y, z) are coordinates of any point on the line.

If l, m and n are the direction cosines of the line, then equation of the line is

$$\boxed{\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}}$$

The cartesian equation of a line passing through origin and having direction ratios (a, b, c) is $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

Method to Convert Vector Equation of Line in Cartesian Form Suppose vector equation of line $\vec{r} = \vec{a} + \lambda \vec{b}$ is given, then to convert it into cartesian form. First, put the values of $\vec{a}$, $\vec{b}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Then, compare the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ from both sides, to get required equation in cartesian form.

Note To find the coordinates of any point on the line, consider $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda$, where λ is some real number

$\Rightarrow$ $x = a\lambda + x_1$, $y = b\lambda + y_1$ and $z = c\lambda + z_1$. Then, $(x = x_1 + a\lambda, y = y_1 + b\lambda, z = z_1 + c\lambda)$ represent the coordinates of any point on the line.

EXAMPLE |1| Find the vector and the cartesian equations of the line passing through the point (5, 2, − 4) and which is parallel to the vector $5\hat{i} + \hat{j} - 7\hat{k}$.

Sol. Given point is (5, 2, −4), whose position vector will be

$\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and parallel vector is $\vec{b} = 5\hat{i} + \hat{j} - 7\hat{k}$

The required vector equation of line is

$$\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(5\hat{i} + \hat{j} - 7\hat{k})$$

For cartesian equation, write

$$\vec{r} = (5 + 5\lambda)\hat{i} + (2 + \lambda)\hat{j} - (4 + 7\lambda)\hat{k}$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = (5 + 5\lambda)\hat{i} + (2 + \lambda)\hat{j} - (4 + 7\lambda)\hat{k}$$

On comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = 5 + 5\lambda,\ y = 2 + \lambda \text{ and } z = -(4 + 7\lambda)$$

$$\Rightarrow \frac{x - 5}{5} = \lambda,\ \frac{y - 2}{1} = \lambda \text{ and } \frac{z + 4}{-7} = \lambda$$

$$\therefore \frac{x - 5}{5} = \frac{y - 2}{1} = \frac{z + 4}{-7} = \lambda$$

Thus, the required cartesian equation of the given line is

$$\frac{x - 5}{5} = \frac{y - 2}{1} = \frac{z + 4}{-7}.$$

Alternate Method

As the given line is parallel to the vector $5\hat{i} + \hat{j} - 7\hat{k}$, therefore direction ratios of the given line are 5, 1, − 7.

Now, the equation of line passing through the point $(5, 2, -4)$ and having direction ratios 5, 1, − 7 is

$$\frac{x - 5}{5} = \frac{y - 2}{1} = \frac{z + 4}{-7}.$$

Note In rectangular coordinate system as X, Y and Z-axes passes through the origin, therefore

Equation of X-axis is $\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0}$

[$\because$ direction cosines of X-axis are (1, 0, 0)]

$\Rightarrow$ $y = 0$ and $z = 0$

Equation of Y-axis is $\frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0} \Rightarrow x = 0$ and $z = 0$

and equation of Z-axis is $\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1} \Rightarrow x = 0$ and $y = 0$.

EXAMPLE |2| Find the vector and cartesian equations of the line through the point (1, 2, − 4) and perpendicular to the two lines

$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$ and

$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$. **[Delhi 2016]**

Sol. Given equations of lines are

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$$

On comparing with

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, we get

$\vec{b_1} = 3\hat{i} - 16\hat{j} + 7\hat{k}$ and $\vec{b_2} = 3\hat{i} + 8\hat{j} - 5\hat{k}$

Now, we determine

$$\vec{b} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$$= \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48)$$

$$= 24\hat{i} + 36\hat{j} + 72\hat{k} = 12(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Since, the required line is perpendicular to the given lines. So, it is parallel to $\vec{b_1} \times \vec{b_2}$. Now, equation of a line passing through the point (1, 2, − 4) and parallel to $24\hat{i} + 36\hat{j} + 72\hat{k}$ or $(2\hat{i} + 3\hat{j} + 6\hat{k})$ is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

which is required vector equation of the line.

For cartesian equation, put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (1 + 2\lambda)\hat{i} + (2 + 3\lambda)\hat{j} + (-4 + 6\lambda)\hat{k}$$

On comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = 1 + 2\lambda,\ y = 2 + 3\lambda \text{ and } z = -4 + 6\lambda$$

$$\Rightarrow \frac{x-1}{2} = \lambda, \frac{y-2}{3} = \lambda \text{ and } \frac{z+4}{6} = \lambda$$

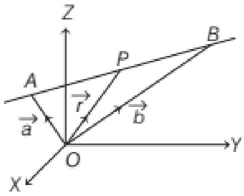
$$\therefore \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

which is the required cartesian equation of the line.

Equation of a Line Passing through Two Given Points

VECTOR EQUATION

The vector equation of a line passing through two points A and B with position vectors $\vec{a}$ and $\vec{b}$ is given by



$$\boxed{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})}$$

where, $\vec{r}$ is the position vector of any point P on the line and λ is some real number.

CARTESIAN EQUATION

The equation of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}}$$

where, (x, y, z) are coordinates of any point on the line.

EXAMPLE |3| If y-coordinate of a point P on the join of $Q(1, 1, 3)$ and $R(2, -3, 5)$ is -4, then find its z-coordinate.

Sol. Equation of a line joining Q and R is

$$\frac{x-1}{2-1} = \frac{y-1}{-3-1} = \frac{z-3}{5-3}$$

$$\Rightarrow \frac{x-1}{1} = \frac{y-1}{-4} = \frac{z-3}{2} \quad \text{...(i)}$$

Since, $P(x, -4, z)$ lies on the line, therefore

$$\frac{x-1}{1} = \frac{-4-1}{-4} = \frac{z-3}{2}$$

Considering last two terms, we get

$$\Rightarrow \frac{z-3}{2} = \frac{-5}{-4} = \frac{5}{4} \Rightarrow z - 3 = \frac{5}{2}$$

$$\Rightarrow z = 3 + \frac{5}{2} = \frac{11}{2}$$

Hence, the required z-coordinate is $\frac{11}{2}$.

Method to Convert Cartesian Equation of a Line in Vector Form First, compare the given equation with $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ to get (x_1, y_1, z_1) and a, b, c, then put these values in $\vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$ and get the required vector equation.

EXAMPLE |4| Find the cartesian and vector equation for the line passing through the points $A(-1, 1, 2)$ and $B(2, 4, 5)$.

Sol. Given points are $A(x_1, y_1, z_1) = (-1, 1, 2)$

and $B(x_2, y_2, z_2) = (2, 4, 5)$

We know that the cartesian equation of a line passing through two points is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\therefore \frac{x-(-1)}{2-(-1)} = \frac{y-1}{4-1} = \frac{z-2}{5-2}$$

$$\Rightarrow \frac{x+1}{3} = \frac{y-1}{3} = \frac{z-2}{3}$$

$$\text{or} \quad \frac{x+1}{1} = \frac{y-1}{1} = \frac{z-2}{1} \quad \text{...(i)}$$

which is the required cartesian equation of the given line.

To find the vector equation of line, compare Eq. (i) with $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$, we get

$$x_1 = -1,\ y_1 = 1,\ z_1 = 2 \text{ and } a = 1, b = 1, c = 1$$

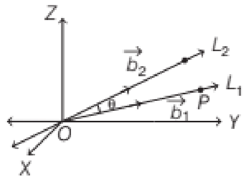
Now, the required vector equation is given by

$$\vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$$

$$\therefore \vec{r} = -\hat{i} + \hat{j} + 2\hat{k} + \lambda(\hat{i} + \hat{j} + \hat{k})$$

Angle between Two Lines

Let L_1 and L_2 be two lines and θ be the acute angle between them.



VECTOR FORM

Let the vector equations of lines L_1 and L_2 be $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, then angle between these two lines is given by

$$\boxed{\cos\theta = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|}$$

where, λ and μ are scalars.

If two lines are perpendicular, then $\vec{b_1} \cdot \vec{b_2} = 0$

and if two lines are parallel, then $\vec{b_1} = \lambda \vec{b_2}$.

CARTESIAN FORM

Let the cartesian equations of lines L_1 and L_2 be

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $$\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}.$$

Then, angle between the lines L_1 and L_2 is given by

$$\cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

The angle between the lines in terms of $\sin\theta$ is given by

$$\sin\theta = \frac{\sqrt{(a_1 b_2 - a_2 b_1)^2 + (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

where, a_1, b_1, c_1 and a_2, b_2, c_2 are direction ratios of lines L_1 and L_2, respectively.

If l_1, m_1, n_1 and l_2, m_2, n_2 are direction cosines of lines L_1 and L_2, then angle between the lines is given by

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

$$[\because l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2]$$

and $$\sin\theta = \sqrt{(l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2}$$

If two lines are perpendicular, then $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

and if two lines are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Note There are always two angles θ and $\pi - \theta$ between two lines.

EXAMPLE |5| Find the angle between the lines

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$

and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$.

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$. **[Foreign 2014]**

Sol. (i) Given lines are $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$

and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

On comparing with $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$,

we get $\vec{b_1} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore$ The angle between the lines is given by

$$\cos\theta = \left|\frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right| = \left|\frac{(3\hat{i} + 2\hat{j} + 6\hat{k})\cdot(\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{3^2 + 2^2 + 6^2}\sqrt{1^2 + 2^2 + 2^2}}\right|$$

$$= \left|\frac{3 + 4 + 12}{\sqrt{9 + 4 + 36}\sqrt{1 + 4 + 4}}\right| = \frac{19}{\sqrt{49}\sqrt{9}}$$

$$= \frac{19}{7 \times 3} = \frac{19}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) Given equations of lines are $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$

and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Here, direction ratios of two lines are (2, 2, 1) and (4, 1, 8).

Let θ be the acute angle between the given lines, then

$$\cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \cos\theta = \frac{|2 \times 4 + 2 \times 1 + 1 \times 8|}{\sqrt{2^2 + 2^2 + 1^2}\sqrt{4^2 + 1^2 + 8^2}}$$

$$= \frac{|8 + 2 + 8|}{\sqrt{4 + 4 + 1}\sqrt{16 + 1 + 64}}$$

$$= \frac{18}{\sqrt{9}\sqrt{81}} = \frac{18}{3 \times 9} = \frac{2}{3}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

EXAMPLE |6| Find the angle between the lines with direction ratios proportional to 4, −3, 5 and 3, 4, 5, respectively.

Sol. Let θ be the acute angle between the given lines.

Here, $a_1 = 4, b_1 = -3, c_1 = 5$ and $a_2 = 3, b_2 = 4, c_2 = 5$

$$\because \cos\theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\therefore \cos\theta = \frac{|4 \times 3 + (-3) \times 4 + 5 \times 5|}{\sqrt{4^2 + (-3)^2 + 5^2}\sqrt{3^2 + 4^2 + 5^2}}$$

$$= \frac{|12 - 12 + 25|}{\sqrt{16 + 9 + 25}\sqrt{9 + 16 + 25}}$$

$$= \frac{25}{\sqrt{50}\sqrt{50}} = \frac{25}{50} = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \cos\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

EXAMPLE |7| Find the angle between the lines whose direction cosines are given by the equations $3l + m + 5n = 0$, $6mn - 2nl + 5lm = 0$. **[NCERT Exemplar]**

Sol. The given equations are

$$3l + m + 5n = 0 \quad \text{...(i)}$$

and $$6mn - 2nl + 5lm = 0 \quad \text{...(ii)}$$

Now, from Eq. (i), we get $m = -3l - 5n$ ...(iii)

On substituting $m = -3l - 5n$ in Eq. (ii), we get

$$6(-3l - 5n)n - 2nl + 5l(-3l - 5n) = 0$$

$\Rightarrow$ $30n^2 + 45ln + 15l^2 = 0 \Rightarrow 2n^2 + 3ln + l^2 = 0$

$\Rightarrow$ $2n^2 + 2nl + nl + l^2 = 0$

$\Rightarrow$ $2n(n + l) + l(n + l) = 0$

$\Rightarrow$ $(n + l)(2n + l) = 0$

Either $l = -n$ or $l = -2n$

If $l = -n$, then $m = -2n$ [using Eq. (iii)]

and if $l = -2n$, then $m = n$. [using Eq. (iii)]

Thus, the direction ratios of two lines are proportional to $(-n, -2n, n)$ and $(-2n, n, n)$ i.e. $(-1, -2, 1)$ and $(-2, 1, 1)$, respectively.

Now, let θ be the acute angle between the lines, then

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{|2 - 2 + 1|}{\sqrt{1 + 4 + 1}\sqrt{4 + 1 + 1}} = \frac{1}{6}$$

$\Rightarrow$ $$\theta = \cos^{-1}\left(\frac{1}{6}\right)$$

Point of Intersection of Lines

To check whether the two given lines intersect or not and to find point of intersection (if intersect) we follow the following procedure

VECTOR FORM

Let the two lines be

$$\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \quad \text{...(i)}$$

and $$\vec{r} = (a_1'\hat{i} + a_2'\hat{j} + a_3'\hat{k}) + \mu(b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}) \quad \text{...(ii)}$$

If Eqs. (i) and (ii) intersect, then they have a common point.

So, we have $(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) + \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$$= (a_1'\hat{i} + a_2'\hat{j} + a_3'\hat{k}) + \mu(b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k})$$

$\Rightarrow$ $(a_1 + \lambda b_1)\hat{i} + (a_2 + \lambda b_2)\hat{j} + (a_3 + \lambda b_3)\hat{k}$

$$= (a_1' + \mu b_1')\hat{i} + (a_2' + \mu b_2')\hat{j} + (a_3' + \mu b_3')\hat{k}$$

$\therefore$ $a_1 + \lambda b_1 = a_1' + \mu b_1'$, $a_2 + \lambda b_2 = a_2' + \mu b_2'$

and $a_3 + \lambda b_3 = a_3' + \mu b_3'$

Now, find the value of λ and μ by solving any two of above equations. If the values of λ and μ satisfy the third equation, then the two lines intersect, otherwise not. If intersect, then the point of intersection can be obtain by substituting the value of λ (or μ) in Eq. (i) [or Eq. (ii)].

EXAMPLE |8| Show that lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ intersect each other. Find their point of intersection. **[Delhi 2014]**

First, determine the values of λ and μ by equating $\vec{r}$ of both the lines and then find $\vec{r}$ by using the value of λ or μ.

Sol. Given lines are $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ ...(i)

and $$\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k}) \quad \text{...(ii)}$$

Clearly, these lines will intersect, if $(\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$, for some particular values of λ and μ.

Collecting the like terms, we get

$$(1 + 3\lambda)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} = (4 + 2\mu)\hat{i} + (-1 + 3\mu)\hat{k}$$

On equating the coefficients of $\hat{i}$, we get

$$1 + 3\lambda = 4 + 2\mu \Rightarrow 3\lambda - 2\mu = 3$$

On equating the coefficients of $\hat{j}$, we get

$$1 - \lambda = 0 \Rightarrow \lambda = 1$$

On equating the coefficients of $\hat{k}$, we get

$$-1 = -1 + 3\mu \Rightarrow \mu = 0$$

$\therefore$ $\mu = 0$ and $\lambda = 1$

Also, these values satisfy $3\lambda - 2\mu = 3$, therefore the given lines intersect.

On putting $\lambda = 1$ in Eq. (i), we get

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + 1(3\hat{i} - \hat{j}) = 4\hat{i} + 0\hat{j} - \hat{k}$$

Hence, the point of intersection of given lines is $(4, 0, -1)$.

CARTESIAN FORM

Let two lines be $L_1 : \dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1} = \lambda$ (say)

and $L_2 : \dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2} = \mu$ (say)

Consider the coordinates of general points on L_1 and L_2, i.e. $(x_1 + a_1\lambda, y_1 + b_1\lambda, z_1 + c_1\lambda)$ ...(i)

and $(x_2 + a_2\mu, y_2 + b_2\mu, z_2 + c_2\mu)$ ...(ii)

where, λ and μ are some real constants. If the lines L_1 and L_2 intersect, then they have a common point.

$\therefore$ $$(x_1 + a_1\lambda, y_1 + b_1\lambda, z_1 + c_1\lambda) = (x_2 + a_2\mu, y_2 + b_2\mu, z_2 + c_2\mu)$$

for some constants λ and μ.

$$\Rightarrow \quad x_1 + a_1\lambda = x_2 + a_2\mu,$$
$$y_1 + b_1\lambda = y_2 + b_2\mu$$
and
$$z_1 + c_1\lambda = z_2 + c_2\mu$$

Now, find the value of λ and μ by solving any two of above equations. If the values of λ and μ satisfy the third equation, then the two lines intersect, otherwise not. If intersect, then the point of intersection can be obtain by substituting the value of λ(or μ) in Eq. (i) [or Eq. (ii)].

EXAMPLE |9| Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also, find their point of intersection. **[Delhi 2020]**

Sol. The given lines are

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = \lambda \text{ (let)} \quad \text{...(i)}$$

and $$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} = \mu \text{ (let)} \quad \text{...(ii)}$$

Then, any point on line (i) is

$$P(3\lambda - 1, 5\lambda - 3, 7\lambda - 5) \quad \text{...(iii)}$$

and any point on line (ii) is

$$Q(\mu + 2, 3\mu + 4, 5\mu + 6) \quad \text{...(iv)}$$

Clearly, the lines (i) and (ii) will intersect, if

$$(3\lambda - 1, 5\lambda - 3, 7\lambda - 5) = (\mu + 2, 3\mu + 4, 5\mu + 6),$$

for some particular value of λ and μ.

$\Rightarrow 3\lambda - 1 = \mu + 2, 5\lambda - 3 = 3\mu + 4$ and $7\lambda - 5 = 5\mu + 6$

$$\Rightarrow \quad 3\lambda - \mu = 3, \quad \text{...(v)}$$
$$5\lambda - 3\mu = 7 \quad \text{...(vi)}$$
and
$$7\lambda - 5\mu = 11 \quad \text{...(vii)}$$

On multiplying Eq. (v) by 3 and then subtracting Eq. (vi) from it, we get

$$9\lambda - 3\mu - 5\lambda + 3\mu = 9 - 7$$

$$\Rightarrow \quad 4\lambda = 2 \Rightarrow \lambda = \frac{1}{2}$$

On putting the value of λ in Eq. (v), we get

$$3 \times \frac{1}{2} - \mu = 3 \Rightarrow \frac{3}{2} - \mu = 3 \Rightarrow \mu = -\frac{3}{2}$$

On putting the values of λ and μ in Eq. (vii), we get

$$7 \times \frac{1}{2} - 5\left(-\frac{3}{2}\right) = 11 \Rightarrow \frac{7}{2} + \frac{15}{2} = 11 \Rightarrow \frac{22}{2} = 11$$

$\Rightarrow 11 = 11$, which is true.

Hence, lines (i) and (ii) intersect and their point of intersection is

$$P\left(3 \times \frac{1}{2} - 1, 5 \times \frac{1}{2} - 3, 7 \times \frac{1}{2} - 5\right) \left[\text{putting } \lambda = \frac{1}{2} \text{ in Eq.(iii)}\right]$$

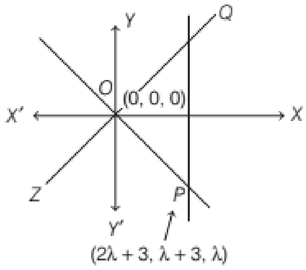
i.e. $P\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

EXAMPLE |10| Find the equations of the two lines passing through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each. **[NCERT Exemplar]**

Sol. Given equation of the line is

$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} = \lambda \text{ (say)}$$

$$\Rightarrow \quad x = 2\lambda + 3, y = \lambda + 3 \text{ and } z = \lambda \quad \text{...(i)}$$



Since, direction ratios of the given line are (2, 1, 1) and the required lines make angle $\frac{\pi}{3}$ with the given line.

$$\therefore \quad \cos\frac{\pi}{3} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\therefore \quad \cos\frac{\pi}{3} = \frac{2(2\lambda + 3) + 1\cdot(\lambda + 3) + 1\cdot\lambda}{\sqrt{2^2 + 1^2 + 1^2}\sqrt{(2\lambda + 3)^2 + (\lambda + 3)^2 + \lambda^2}}$$

[$\because$ direction ratios of OP are $(2\lambda + 3), (\lambda + 3)$ and λ]

$$\Rightarrow \quad \frac{1}{2} = \frac{6\lambda + 9}{\sqrt{6}\sqrt{4\lambda^2 + 9 + 12\lambda + \lambda^2 + 9 + 6\lambda + \lambda^2}}$$

$$\Rightarrow \quad \frac{\sqrt{6}}{2} = \frac{6\lambda + 9}{\sqrt{6\lambda^2 + 18\lambda + 18}}$$

$$\Rightarrow \quad 6\sqrt{(\lambda^2 + 3\lambda + 3)} = 2(6\lambda + 9)$$

$$\Rightarrow \quad 6\sqrt{(\lambda^2 + 3\lambda + 3)} = 6(2\lambda + 3)$$

$$\Rightarrow \quad (\lambda^2 + 3\lambda + 3) = (4\lambda^2 + 9 + 12\lambda)$$

[squaring on both sides]

$$\Rightarrow \quad 3\lambda^2 + 9\lambda + 6 = 0$$

$$\Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow \quad (\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = -1, -2$$

So, the direction ratios of required lines are $(1, 2, -1)$ and $(-1, 1, -2)$. [putting $\lambda = -1, -2$ in DR's of OP]

Since, the required lines passes through origin.

$\therefore$ The equations of required lines are $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$.

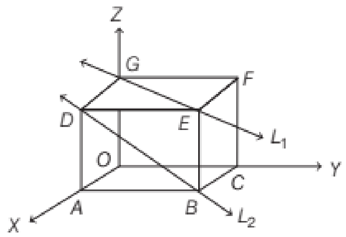
SHORTEST DISTANCE BETWEEN TWO LINES

Two lines in space will be one of the following forms

(i) Intersecting (ii) Parallel

(iii) Neither parallel nor intersecting

If two lines in space intersect at a point, then the shortest distance between them is zero. If two lines in space are parallel, then the shortest distance between them will be the perpendicular distance, i.e. the length of the perpendicular drawn from a point on one line onto the other line.



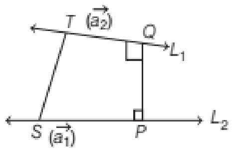
If two lines are neither intersecting nor parallel, then such pair of lines are **non-coplanar** and are called **skew-lines**. In the given figure, line *GE* (lie in ceiling *DEFG*) and *BD* (lie in wall *ABED*) are skew-lines, since they are not parallel and also never meet.

Note Two lines lying in the same plane are called **coplanar lines**. Coplanar lines are either parallel or intersecting.

Shortest Distance between Two Skew-Lines

For skew-lines, the line of the shortest distance will be perpendicular to both the lines and it is unique also.

In figure, the shortest distance (SD) between two skew-lines L_1 and L_2 is the length of the line segment PQ.



VECTOR FORM

Let the equations of L_1 and L_2 be

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \quad \text{and} \quad \vec{r} = \vec{a_2} + \mu \vec{b_2}$$

Then, shortest distance *PQ* between these two skew-lines is

$$\text{SD} = \frac{\left|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})\right|}{|\vec{b_1} \times \vec{b_2}|}$$

Condition for Two Given Lines to be Intersect The given lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ intersect, if the shortest distance between them is zero.

i.e. $$\frac{\left|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})\right|}{|\vec{b_1} \times \vec{b_2}|} = 0$$

$$\Rightarrow \quad \boxed{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0}$$

Method to Find the Shortest Distance Let two lines be $\vec{r} = \vec{a_1} + \vec{b_1}\lambda$ and $\vec{r} = \vec{a_2} + \vec{b_2}\mu$, which are the standard equations of lines.

Then, for finding the shortest distance between the two lines, use the following steps

I. First, check whether the given equations are in standard form or not. If they are not in standard form, then write them in standard form.

II. Find $\vec{a_1}, \vec{a_2}, \vec{b_1}$ and $\vec{b_2}$ by comparing with standard form, say

i.e. $\vec{a_1} = a_1'\hat{i} + a_2'\hat{j} + a_3'\hat{k}$, $\vec{b_1} = b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}$

and $\vec{a_2} = a_1''\hat{i} + a_2''\hat{j} + a_3''\hat{k}$, $\vec{b_2} = b_1''\hat{i} + b_2''\hat{j} + b_3''\hat{k}$.

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1' & b_2' & b_3' \\ b_1'' & b_2'' & b_3'' \end{vmatrix}$$

$$= \hat{i}(b_2'b_3'' - b_3'b_2'') - \hat{j}(b_1'b_3'' - b_3'b_1'') + \hat{k}(b_1'b_2'' - b_2'b_1'')$$

IV. Determine the value of $\vec{a_2} - \vec{a_1}$ and $|\vec{b_1} \times \vec{b_2}|$.

V. Now, put the values obtained in steps III and IV in shortest distance formula

i.e. $$\text{SD} = \frac{\left|(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})\right|}{|\vec{b_1} \times \vec{b_2}|}$$

and simplify it to get the required shortest distance.

EXAMPLE |11| Find the shortest distance between the lines $\vec{r} = (t+1)\hat{i} + (2-t)\hat{j} + (1+t)3\hat{k}$ and $\vec{r} = (2s+2)\hat{i} - (1-s)\hat{j} + (2s-1)3\hat{k}$. **[Delhi 2016C]**

Sol. Given lines are

$$\vec{r} = (t+1)\hat{i} + (2-t)\hat{j} + (1+t)3\hat{k}$$

and $\vec{r} = (2s+2)\hat{i} - (1-s)\hat{j} + (2s-1)3\hat{k}$

Clearly, the given equations are not in standard form. Let us write these equations in standard form, i.e.

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + t(\hat{i} - \hat{j} + 3\hat{k}) \quad \text{... (i)}$$

$$\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + s(2\hat{i} + \hat{j} + 6\hat{k}) \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii) with $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda \vec{b}_2$, we get

Let $\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b}_1 = \hat{i} - \hat{j} + 3\hat{k}$,

$\vec{a}_2 = 2\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b}_2 = 2\hat{i} + \hat{j} + 6\hat{k}$.

Then, $\vec{a}_2 - \vec{a}_1 = \hat{i} - 3\hat{j} - 6\hat{k}$

We know that $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & 1 & 6 \end{vmatrix} = -9\hat{i} + 3\hat{k}$

$$\therefore \text{Shortest distance} = \left| \frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

$$= \left| \frac{-9-18}{\sqrt{81+9}} \right| = \frac{27}{3\sqrt{10}} = \frac{9\sqrt{10}}{10}$$

EXAMPLE |12| Find the shortest distance between the lines $\vec{r} = (1+\lambda)\hat{i} + (2-3\lambda)\hat{j} + (3+2\lambda)\hat{k}$ and $\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$. **[NCERT]**

Sol. Given lines are

$$\vec{r} = (1+\lambda)\hat{i} + (2-3\lambda)\hat{j} + (3+2\lambda)\hat{k} \quad \text{...(i)}$$

and $\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$. ...(ii)

Clearly, first equation is not in standard form. Let us write first equation of line in standard form, i.e.

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \quad \text{...(iii)}$$

On comparing Eqs. (iii) and (ii) with $\vec{r} = a_1 + \lambda b_1$ and $\vec{r} = a_2 + \lambda b_2$ respectively, we get

$$\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}, \quad \vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$$

and $\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}$, $\vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$

Clearly, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

$$= \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6)$$

$$= -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \quad |\vec{b_1} \times \vec{b_2}| = |-9\hat{i} + 3\hat{j} + 9\hat{k}| = \sqrt{(-9)^2 + (3)^2 + (9)^2}$$

$$= \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

Now, $\vec{a_2} - \vec{a_1} = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\text{Required SD} = \frac{|(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

$$= \frac{|(3\hat{i} + 3\hat{j} + 3\hat{k})\cdot(-9\hat{i} + 3\hat{j} + 9\hat{k})|}{3\sqrt{19}}$$

$$= \frac{|-27+9+27|}{3\sqrt{19}} = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}} \text{ units}$$

which is the required shortest distance.

CARTESIAN FORM

Let L_1 and L_2 be two skew-lines with equations

$$L_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $$L_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

Then, the shortest distance between these lines is

$$\left| \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}} \right|$$

Condition for Two Given Lines to be Intersect Let the two lines be

$$L_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $$L_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}.$$

The lines L_1 and L_2 will intersect, if the shortest distance between them is zero

i.e. $$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Note If direction cosines of the lines are given, i.e. (l_1, m_1, n_1) and (l_2, m_2, n_2), then replace (a_1, b_1, c_1) by (l_1, m_1, n_1) and (a_2, b_2, c_2) by (l_2, m_2, n_2) in the above formula to find shortest distance.

Method to Find the Shortest Distance Suppose two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are given to us, then for finding the shortest distance between these lines, use the following steps

I. First, check whether the given equations are in standard form or not. If they are not in standard form, then write them in standard form.

II. Find $x_1, y_1, z_1; x_2, y_2, z_2; a_1, b_1, c_1$ and a_2, b_2, c_2 by comparing the given equations with standard equation of lines.

III. Compute the value of the determinant

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

by expanding along R_1 or C_1.

IV. Compute the value of

$$\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$$

V. Put the values obtained in steps III and IV in shortest distance formula, i.e.

$$\text{SD} = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

and simplify it to get required shortest distance.

EXAMPLE |13| Find the shortest distance between the lines $\frac{1-x}{-2} = \frac{2-y}{-3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$.

Sol. Given lines are $\frac{1-x}{-2} = \frac{2-y}{-3} = \frac{z-3}{4}$

and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$, which are not in standard form.

Given equations of lines can be written in standard form as

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

On comparing the given equations of lines with

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2},$$

we get $x_1 = 1, y_1 = 2, z_1 = 3; a_1 = 2, b_1 = 3, c_1 = 4$

and $x_2 = 2, y_2 = 4, z_2 = 5; a_2 = 3, b_2 = 4, c_2 = 5$

On putting these values in $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$,

we get $\begin{vmatrix} 2-1 & 4-2 & 5-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$$= 1(15 - 16) - 2(10 - 12) + 2(8 - 9) = -1 + 4 - 2 = 1$$

Now, $\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}$

$$= \sqrt{(3 \times 5 - 4 \times 4)^2 + (4 \times 3 - 5 \times 2)^2 + (2 \times 4 - 3 \times 3)^2}$$

$$= \sqrt{(15 - 16)^2 + (12 - 10)^2 + (8 - 9)^2}$$

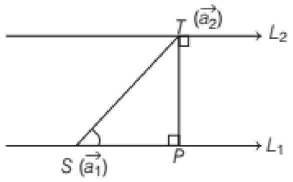
$$= \sqrt{(-1)^2 + (2)^2 + (-1)^2} = \sqrt{1 + 4 + 1} = \sqrt{6}$$

$$\therefore \quad \text{SD} = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}},$$

$= \frac{1}{\sqrt{6}}$ units, which is the required shortest distance.

Distance between Parallel Lines

If two lines L_1 and L_2 are parallel, then they are coplanar.



The shortest distance TP between parallel lines

$L_1 : \vec{r} = \vec{a_1} + \lambda \vec{b}$ and $L_2 : \vec{r} = \vec{a_2} + \mu \vec{b}$ is

$$\text{SD or } d = |\overrightarrow{PT}| = \left| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right|$$

Note To find the distance between two lines, first check whether they are parallel or not.

EXAMPLE |14| Find the distance between the lines L_1 and L_2 given by $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$. **[Foreign 2014]**

Sol. Given lines are

$$L_1 : \vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $L_2 : \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$

or $L_2 : \vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 6\hat{k})$

$$= (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu'(2\hat{i} + 3\hat{j} + 6\hat{k})$$

On comparing the given equations of lines with $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu' \vec{b}$, we get $\vec{a_1} = \hat{i} + 2\hat{j} - 4\hat{k}$,

$\vec{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

Now, $\vec{a_2} - \vec{a_1} = (3\hat{i} + 3\hat{j} - 5\hat{k}) - (\hat{i} + 2\hat{j} - 4\hat{k}) = 2\hat{i} + \hat{j} - \hat{k}$

and $\vec{b} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$

$$= (-3-6)\hat{i} - (-2-12)\hat{j} + (2-6)\hat{k}$$
$$= -9\hat{i} + 14\hat{j} - 4\hat{k}$$

Required distance, $d = \left| \dfrac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right| = \dfrac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{|2\hat{i} + 3\hat{j} + 6\hat{k}|}$

$$= \frac{\sqrt{(-9)^2 + (14)^2 + (-4)^2}}{\sqrt{(2)^2 + (3)^2 + (6)^2}}$$

$$= \frac{\sqrt{81+196+16}}{\sqrt{4+9+36}} = \frac{\sqrt{293}}{\sqrt{49}}$$

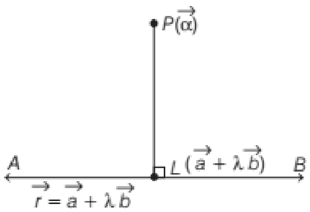
$$= \frac{\sqrt{293}}{7} \text{ units}$$

Perpendicular Distance of a Line from a Given Point

VECTOR FORM

To determine the perpendicular distance from a given point $P(\vec{\alpha})$ to a given line $\vec{r} = \vec{a} + \lambda \vec{b}$ we follow the following procedure

Let L be the foot of perpendicular drawn from the point $P(\vec{\alpha})$ on the given line and the position vector of L be $\vec{a} + \lambda \vec{b}$.



Then, $\vec{PL} = \vec{a} + \lambda \vec{b} - \vec{\alpha} = \vec{a} - \vec{\alpha} + \lambda \vec{b}$.

Since, $\vec{PL}$ is perpendicular to the line which is parallel to $\vec{b}$.

Therefore, $\vec{PL} \cdot \vec{b} = 0$

$\Rightarrow \quad (\vec{a} - \vec{\alpha} + \lambda \vec{b}) \cdot \vec{b} = 0$

$\Rightarrow (\vec{a} - \vec{\alpha}) \cdot \vec{b} + \lambda (\vec{b} \cdot \vec{b}) = 0 \Rightarrow \lambda = -\dfrac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}$

On substituting the value of λ in $\vec{a} + \lambda \vec{b}$ and $\vec{PL} = \vec{a} - \vec{\alpha} + \lambda \vec{b}$, we obtain the position vector of L and vector $\vec{PL}$. The magnitude of $\vec{PL}$ gives the length of perpendicular.

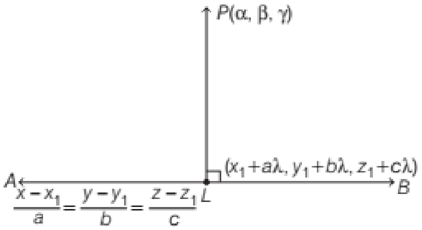
CARTESIAN FORM

To determine the perpendicular distance from a given point $P(\alpha, \beta, \gamma)$ to a given line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \lambda \text{ (say)}$$

we follow the following procedure

Let L be the foot of the perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the given line and the coordinate of L be $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$.



Then, direction ratio of AB are proportional to a, b, c and direction ratio of PL are proportional to $(x_1 + a\lambda - \alpha, \quad y_1 + b\lambda - \beta, \quad z_1 + c\lambda - \gamma)$. Since, PL is perpendicular to AB, therefore

$(x_1 + a\lambda - \alpha)a + (y_1 + b\lambda - \beta)b + (z_1 + c\lambda - \gamma)c = 0.$

After simplifying, we get

$$\lambda = \frac{[a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)]}{a^2 + b^2 + c^2}$$

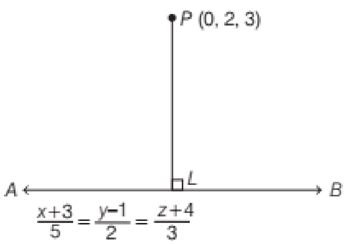
On putting the value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we obtain coordinates of L. Now, we can find the length of PL using distance formula.

EXAMPLE |15| Find the coordinates of foot of perpendicular drawn from the point (0, 2, 3) on the line $\dfrac{x+3}{5} = \dfrac{y-1}{2} = \dfrac{z+4}{3}$. Also, find the length of perpendicular.

Sol. Given equation of the line is $\dfrac{x+3}{5} = \dfrac{y-1}{2} = \dfrac{z+4}{3}$

Let $\dfrac{x+3}{5} = \dfrac{y-1}{2} = \dfrac{z+4}{3} = \lambda$ (say)

$\Rightarrow \dfrac{x+3}{5} = \lambda, \dfrac{y-1}{2} = \lambda$ and $\dfrac{z+4}{3} = \lambda$



$\Rightarrow \quad x = 5\lambda - 3,\ y = 2\lambda + 1$ and $z = 3\lambda - 4$

$\therefore$ Coordinates of point L are $(5\lambda - 3,\ 2\lambda + 1,\ 3\lambda - 4)$

Now, DR's of line PL

$$= (5\lambda - 3 - 0,\ 2\lambda + 1 - 2,\ 3\lambda - 4 - 3)$$
$$= (5\lambda - 3,\ 2\lambda - 1,\ 3\lambda - 7)$$

DR's of line AB are 5, 2, 3.

$\because \quad PL \perp AB$

$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$ ...(i)

where, $a_1 = 5\lambda - 3,\ b_1 = 2\lambda - 1,$

$c_1 = 3\lambda - 7$ and $a_2 = 5,\ b_2 = 2,\ c_2 = 3$

From Eq. (i) we get

$\Rightarrow 5\cdot(5\lambda - 3) + 2\cdot(2\lambda - 1) + 3\cdot(3\lambda - 7) = 0$

$\Rightarrow \quad 25\lambda - 15 + 4\lambda - 2 + 9\lambda - 21 = 0$

$\Rightarrow \quad 38\lambda - 38 = 0 \Rightarrow 38\lambda = 38 \Rightarrow \lambda = 1$

$\therefore$ Foot of perpendicular L

$$= (5\lambda - 3,\ 2\lambda + 1,\ 3\lambda - 4) = (2, 3, -1) \quad [\text{put } \lambda = 1]$$

Also, length of perpendicular,

$PL =$ Distance between points P and L

$$\therefore \quad = \sqrt{(0-2)^2 + (2-3)^2 + (3+1)^2}$$

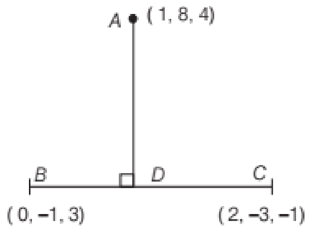
$$[\because \text{distance} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}]$$

$$= \sqrt{4 + 1 + 16} = \sqrt{21} \text{ units}$$

EXAMPLE |16| Find the coordinates of the foot of perpendicular drawn from a point $A(1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$.

[All India 2017C]

Sol. Let D be the foot of perpendicular drawn from A to the line BC.



Now, equation of line passing through B and C is

$$\frac{x}{2} = \frac{y+1}{-2} = \frac{z-3}{-4}$$

or
$$\frac{x}{1} = \frac{y+1}{-1} = \frac{z-3}{-2}$$

Now, let $\dfrac{x}{1} = \dfrac{y+1}{-1} = \dfrac{z-3}{-2} = \lambda$

$\Rightarrow \quad x = \lambda,\ y = -\lambda - 1$ and $z = -2\lambda + 3$

So, coordinates of D are $(\lambda, -\lambda - 1, -2\lambda + 3)$ for some value of λ.

Now, direction ratios of line AD are

$$< \lambda - 1, -\lambda - 9, -2\lambda - 1 >$$

As, $AD \perp BC$

$\therefore \quad 1(\lambda - 1) - 1(-\lambda - 9) - 2(-2\lambda - 1) = 0 \Rightarrow \lambda = -\dfrac{5}{3}$

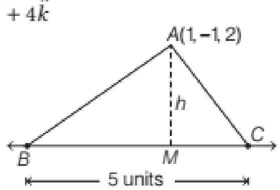
Thus, the coordinates of D is $\left(-\dfrac{5}{3}, \dfrac{2}{3}, \dfrac{19}{3}\right)$.

Hence, the coordinates of foot of perpendicular drawn from A to line joining B and C are $\left(-\dfrac{5}{3}, \dfrac{2}{3}, \dfrac{19}{3}\right)$.

EXAMPLE |17| Vertices B and C of ΔABC lie along the line $\dfrac{x+2}{2} = \dfrac{y-1}{1} = \dfrac{z-0}{4}$. Find the area of the triangle given that A has coordinates $(1, -1, 2)$ and line segment BC has length 5 units.

Sol. Let h be the height of ΔABC. Then, h is the length of perpendicular from $A(1, -1, 2)$ to the line $\dfrac{x+2}{2} = \dfrac{y-1}{1} = \dfrac{z-0}{4}$.

Clearly, line $\dfrac{x+2}{2} = \dfrac{y-1}{1} = \dfrac{z-0}{4}$ passes through the point say $P(-2, 1, 0)$ and parallel to the vector $\vec{b} = 2\hat{i} + \hat{j} + 4\hat{k}$



Let $\dfrac{x+2}{2} = \dfrac{y-1}{1} = \dfrac{z-0}{4} = \lambda$. Then, coordinates of M are $(2\lambda - 2, \lambda + 1, 4\lambda)$.

Now, DR's of AM are $2\lambda - 3,\ \lambda + 2$ and $4\lambda - 2$.

Since, $AM \perp BC$, therefore

$$2(2\lambda - 3) + 1(\lambda + 2) + 4(4\lambda - 2) = 0$$

[$\because$ DR's of line BC are 2, 1, 4]

$\Rightarrow \ 21\lambda = 12 \Rightarrow \lambda = \dfrac{4}{7}$

Thus, the coordinate of M are $\left(\dfrac{-6}{7}, \dfrac{11}{7}, \dfrac{16}{7}\right)$

Now, $h = |AM| = \sqrt{\left(\dfrac{-6}{7} - 1\right)^2 + \left(\dfrac{11}{7} + 1\right)^2 + \left(\dfrac{16}{7} - 2\right)^2}$

$$= \sqrt{\frac{169}{7^2} + \frac{324}{7^2} + \frac{4}{7^2}} = \sqrt{\frac{497}{7^2}} = \sqrt{\frac{71}{7}}$$

It is given that the length of BC is 5 units.

$\therefore$ Area of $\Delta ABC = \dfrac{1}{2}(BC \times h) = \dfrac{1}{2} \times 5 \times \sqrt{\dfrac{71}{7}}$

$$= \sqrt{\frac{1775}{28}} \text{ sq units}$$

TOPIC PRACTICE 2

OBJECTIVE TYPE QUESTIONS

1 The equation of straight line passing through the point (a, b, c) and parallel to Z-axis is

(a) $\frac{x-a}{1}=\frac{y-b}{1}=\frac{z-c}{0}$

(b) $\frac{x-a}{0}=\frac{y-b}{1}=\frac{z-c}{1}$

(c) $\frac{x-a}{1}=\frac{y-b}{0}=\frac{z-c}{0}$

(d) $\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}$

2 The equation of a line passing through the point $(-3, 2, -4)$ and equally inclined to the axes are

(a) $x-3=y+2=z-4$ (b) $x+3=y-2=z+4$

(c) $\frac{x+3}{1}=\frac{y-2}{2}=\frac{z+4}{3}$ (d) None of these

3 The two lines $x=ay+b,\ z=cy+d$ and $x=a'y+b',\ z=c'y+d'$ are perpendicular to each other, if **[Delhi 2020]**

(a) $\frac{a}{a'}+\frac{c}{c'}=1$ (b) $\frac{a}{a'}+\frac{c}{c'}=-1$

(c) $aa'+cc'=1$ (d) $aa'+cc'=-1$

4 The angle between the lines through the points $(4, 7, 8)$, $(2, 3, 4)$ and $(-1, -2, 1)$, $(1, 2, 5)$ is

(a) 0 (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

5 The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{4-z}{k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{-2}$ are mutually perpendicular, if the value of k is **[All India 2020]**

(a) $-\frac{2}{3}$ (b) $\frac{2}{3}$

(c) -2 (d) 2

VERY SHORT ANSWER Type Questions

6 Find the direction cosines of the line

$$\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}.$$ **[Delhi 2013C]**

7 The equation of a line are $5x-3=15y+7=3-10z$. Write the direction cosines of the line. **[All India 2015]**

8 If the equation of line AB is $\frac{3-x}{1}=\frac{y+2}{-2}=\frac{z-5}{4}$, then write the direction ratios of the line parallel to above line AB. **[Delhi 2011]**

9 Find the cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$. **[Delhi 2013]**

10 Find the vector equation of the line which passes through the point $(3, 4, 5)$ and is parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$. **[Delhi 2019]**

11 A line passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction of the vector $\hat{i}+\hat{j}-2\hat{k}$. Find the equation of the line in cartesian form. **[All India 2019]**

12 A line passes through the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ and makes angles 60°, 120° and 45° with X, Y and Z-axes, respectively. Find the equation of the line in the cartesian form. **[Delhi 2016C]**

SHORT ANSWER Type I Questions

13 The vector equation of a line which passes through the points $(3, 4, -7)$ and $(1, -1, 6)$ is .... **[All India 2020]**

14 Find the equation of a line in cartesian form, which is parallel to $2\hat{i}-\hat{j}+3\hat{k}$ and which passes through the point $(5, -2, 4)$.

15 The x-coordinate of a point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find its z-coordinate. **[All India 2017]**

16 Find the cartesian equation of line that passing through the points $(1, -1, 3)$ and $(3, 4, -2)$.

17 Find the vector equation of line passing through the points $(1, -1, 2)$ and $(3, 2, 1)$.

18 Find the vector equation of the line passing through the point $A(1, 2, -1)$ and parallel to the line $5x-25=14-7y=35z$. **[Delhi 2017]**

SHORT ANSWER Type II Questions

19 Find the shortest distance between the lines

$$r=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-3k)$$

and $$r=(\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k}).$$ **[CBSE 2018]**

20 The cartesian equations of a line is $6x-2=3y+1=2z-2$. Find the direction cosines of the line. Write down the cartesian and vector equations of a line passing through the point $(2,-1,-1)$ which are parallel to the given line. **[Delhi 2013C]**

21 Find the direction cosines of the line $\frac{x+2}{2}=\frac{2y-7}{6}=\frac{5-z}{6}$. Also, find the vector equation of the line through the point $A(-1,2,3)$ and parallel to the given line. **[Delhi 2014]**

22 Find the equation of a line passing through the point $(1,2,-4)$ and perpendicular to two lines $\vec{r}=(8\hat{i}-19\hat{j}+10\hat{k})+\lambda(3\hat{i}-16\hat{j}+7\hat{k})$ and $\vec{r}=(15\hat{i}+29\hat{j}+5\hat{k})+\mu(3\hat{i}+8\hat{j}-5\hat{k})$. **[All India 2015]**

23 A line passes through the point $(2,-1,3)$ and is perpendicular to the lines $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$. Obtain its equation in vector and cartesian forms. **[All India 2014]**

24 Find the angle between the pair of lines given by $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ and $\vec{r}=5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$.

25 Find the angle between the lines $\vec{r}=\lambda(\hat{i}+\hat{j}+2\hat{k})$ and $\vec{r}=2\hat{j}+\mu\{(\sqrt{3}-1)\hat{i}-(\sqrt{3}+1)\hat{j}+4\hat{k}\}$.

26 Find the angle between the pair of lines $\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$ and check whether the lines are parallel or perpendicular. **[Delhi 2011]**

27 Find the angle between the lines $2x=3y=-z$ and $6x=-y=-4z$.

28 Find the value of p so that the lines $\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{1}$ and $\frac{7-7x}{3p}=\frac{5-y}{1}=\frac{11-z}{7}$ are at right angles. **[Delhi 2017C]**

29 Show that the line through the points $(1,-1,2)$, $(3,4,-2)$ is perpendicular to the line through the points $(0,3,2)$ and $(3,5,6)$. **[NCERT]**

30 Find the vector and cartesian equations of the line which is perpendicular to the lines with equations $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and passes through the point $(1,1,1)$. Also, find the angle between the given lines. **[All India 2020]**

31 Find the vector and cartesian equation of a line through the point $(1,-1,1)$ and perpendicular to the lines joining the points $(4,3,2)$, $(1,-1,0)$ and $(1,2,-1)$, $(2,1,1)$.

32 Find the shortest distance between the lines $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and $\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$. **[All India 2015C]**

33 Find the shortest distance between the lines $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$. **[Foreign 2014]**

34 By computing shortest distance, determine whether the following pair of lines intersect or not $\vec{r}=(4\hat{i}+5\hat{j})+\lambda(\hat{i}+2\hat{j}-3\hat{k})$ and $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k})$.

35 Show that the lines $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ and $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ do not intersect each other.

36 Find the coordinates of the foot of perpendicular drawn from the point $(2,3,-8)$ to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. **[All India 2017C]**

37 Find the foot of perpendicular from $P(1,2,-3)$ to the line $\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}$. Also, find the image of P in the given line. **[Delhi 2016C]**

38 Find the coordinates of the foot of perpendicular drawn from the point $A(-1,8,4)$ to the line joining the points $B(0,-1,3)$ and $C(2,-3,-1)$. Hence, find the image of the point A in the line BC. **[All India 2016]**

LONG ANSWER Type Questions

39 Find the vector and cartesian equations of a line passing through (1, 2, – 4) and perpendicular to the two lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. **[Delhi 2017]**

40 If the lines $\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$ and $\frac{x-1}{3\lambda}=\frac{y-1}{2}=\frac{z-6}{-5}$ are perpendicular, find the value of λ. Hence find whether the lines are intersecting or not. **[All India 2019]**

41 Show that the lines $\frac{x-1}{3}=\frac{y-1}{-1}, z+1=0$ and $\frac{x-4}{2}=\frac{z+1}{3}, y=0$ intersect each other. Also, find their point of intersection.

42 Find the shortest distance between the lines $\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$.
Also, find the equations of the shortest distance.

43 Find the perpendicular distance of point (1, 0, 0) from the line $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$. Also, find the coordinate of foot of perpendicular and equation of perpendicular. **[Delhi 2011]**

HINTS & SOLUTIONS

1. (d) **Hint** Direction cosines of Z-axis are 0, 0, 1.

2. (b) Here, $l = m = n$, therefore required equation of line is

$$\frac{x+3}{l}=\frac{y-2}{l}=\frac{z+4}{l}$$
$$\Rightarrow x+3=y-2=z+4$$

3. (d) We have,

$$x = ay+b,\ z = cy+d \text{ and } x = a'y+b',\ z = c'y+d'$$
$$\Rightarrow \frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c} \text{ and } \frac{x-b'}{a'}=\frac{y}{1}=\frac{z-d'}{c'}$$

Since, these lines are perpendicular.

$$\therefore \quad aa'+1+cc'=0$$

[∵ two lines are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$]

$$\Rightarrow \quad aa'+cc'=-1$$

4. (a) DR's of given line are –2, – 4, – 4 and 2, 4, 4.

Since, DR's are proportional, therefore given lines are parallel to each other.

Hence, angle between them is zero.

5. (a) We have,

$$\frac{x-2}{1}=\frac{y-3}{1}=\frac{4-z}{k} \text{ and } \frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{-2}$$

or $$\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k} \text{ and } \frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{-2}$$

Since, the given lines are perpendicular.

$$\therefore \quad (1)(k)+(1)(2)+(-k)(-2)=0$$
$$\Rightarrow \quad k+2+2k=0$$
$$\Rightarrow \quad 3k+2=0 \Rightarrow k=-\frac{2}{3}$$

6. Given equation of line is $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$.

It can be rewritten in standard form as $\frac{x-4}{-2}=\frac{y}{6}=\frac{z-1}{-3}$.

Here, DR's of line are –2, 6, – 3.

Now, $\sqrt{(-2)^2+6^2+(-3)^2}=\sqrt{4+36+9}=\sqrt{49}=7$

So, DC's of line are $-\frac{2}{7}, \frac{6}{7}, -\frac{3}{7}$.

7. Given equation of a line is

$$5x-3=15y+7=3-10z \quad \text{...(i)}$$

To convert the equation in standard form

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \quad \text{...(ii)}$$

Let us divide Eq. (i) by LCM (coefficient of x, y and z).
i.e. LCM (5, 15, 10) = 30

Now, Eq. (i) becomes $\frac{5x-3}{30}=\frac{15y+7}{30}=\frac{3-10z}{30}$

$$\Rightarrow \frac{5\left(x-\frac{3}{5}\right)}{30}=\frac{15\left(y+\frac{7}{15}\right)}{30}=\frac{-10\left(z-\frac{3}{10}\right)}{30}$$

$$\Rightarrow \frac{x-\frac{3}{5}}{6}=\frac{y+\frac{7}{15}}{2}=\frac{z-\frac{3}{10}}{-3}$$

On comparing the above equation with Eq. (ii), we get 6, 2, – 3 are the direction ratios of the given line.
Now, the direction cosines of given line are

$$\frac{6}{\sqrt{6^2+2^2+(-3)^2}}, \frac{2}{\sqrt{6^2+2^2+(-3)^2}}, \frac{-3}{\sqrt{6^2+2^2+(-3)^2}}$$

i.e. $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$ $\quad$ [∵ $\sqrt{36+4+9}=\sqrt{49}=7$]

8. **Hint** Given equation can be rewritten in standard form as

$$\frac{x-3}{-1}=\frac{y+2}{-2}=\frac{z-5}{4}$$

(i) Direction ratios of two parallel lines are proportional
(ii) Direction ratios of given line are –1, – 2, 4 .
[Ans. –1, –2, 4]

9. Since, the required line is parallel to the line

$$\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$$

or $$\frac{x+3}{3}=\frac{y-4}{-5}=\frac{z+8}{6}$$

∴ DR's of both lines are proportional to each other.

The required cartesian equation of the line passing through the point (−2, 4, −5) having DR's (3, −5, 6) is

$$\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}$$

10. Equation of a line passing through a point with position vector $\vec{a}$ and parallel to a vector $\vec{b}$ is

$$\vec{r}=\vec{a}+\lambda\vec{b}$$

Since, line passes through (3, 4, 5)

$$\therefore \quad \vec{a}=3\hat{i}+4\hat{j}+5\hat{k}$$

Since, line is parallel to $2\hat{i}+2\hat{j}-3\hat{k}$

$$\therefore \quad \vec{b}=2\hat{i}+2\hat{j}-3\hat{k}$$

Equation of line is $\vec{r}=\vec{a}+\lambda\vec{b}$,

i.e. $\vec{r}=(3\hat{i}+4\hat{j}+5\hat{k})+\lambda(2\hat{i}+2\hat{j}-3\hat{k})$,

which is the required vector equation.

11. The given line passes through the point A having position vector $\vec{a}=2\hat{i}-\hat{j}+4\hat{k}$ and is parallel to the vector $\vec{b}=(\hat{i}+\hat{j}-2\hat{k})$

∴ The equation of the given line is

$$\vec{r}=\vec{a}+\lambda\vec{b} \Rightarrow \vec{r}=(2\hat{i}-\hat{j}+4\hat{k})+\lambda(\hat{i}+\hat{j}-2\hat{k}) \quad \text{...(i)}$$

For cartesian equation, put $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in Eq. (i), we get

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})=(2\hat{i}-\hat{j}+4\hat{k})+\lambda(\hat{i}+\hat{j}-2\hat{k})$$

$$\Rightarrow x\hat{i}+y\hat{j}+z\hat{k}=(2+\lambda)\hat{i}+(\lambda-1)\hat{j}+(4-2\lambda)\hat{k}$$

$$\Rightarrow x=2+\lambda,\ y=\lambda-1 \text{ and } z=4-2\lambda$$

$$\Rightarrow \frac{x-2}{1}=\frac{y+1}{1}=\frac{z-4}{-2}=\lambda$$

Hence, $\frac{x-2}{1}=\frac{y+1}{1}=\frac{z-4}{-2}$ is the required equation of the given line in cartesian form.

12. **Hint** DC's of required line are cos 60°, cos 120° and cos 45°; i.e. $\frac{1}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}}$ and a point on the line is (2, −3, 4).

[Ans. $\frac{x-2}{1/2}=\frac{y+3}{-1/2}=\frac{z-4}{1/\sqrt{2}}$

or $2x-4=-2y-6=\sqrt{2}(z-4)$]

13. Any line passing through the points $\vec{a}$ and $\vec{b}$ has vector equation $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$.

∴ Vector equation of the line passing through the points $\vec{a}=3\hat{i}+4\hat{j}-7\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+6\hat{k}$ is given by

$$\vec{r}=3\hat{i}+4\hat{j}-7\hat{k}+\lambda[(\hat{i}-\hat{j}+6\hat{k})-(3\hat{i}+4\hat{j}-7\hat{k})]$$

$$\Rightarrow \vec{r}=3\hat{i}+4\hat{j}-7\hat{k}+\lambda(-2\hat{i}-5\hat{j}+13\hat{k})$$

14. Similar as Example 1. $\left[\text{Ans. } \frac{x-5}{2}=\frac{y+2}{-1}=\frac{z-4}{3}\right]$

15. Similar as Example 3. [**Ans.** $z=-1$]

16. Similar as Example 4. $\left[\text{Ans. } \frac{x-1}{2}=\frac{y+1}{5}=\frac{z-3}{-5}\right]$

17. **Hint** Let $\vec{a}$ and $\vec{b}$ are the position vector of (1, −1, 2) and (3, 2, 1), respectively. Then, $\vec{a}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{j}+\hat{k}$.

Now, the required equation of line is given by $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$.

[**Ans.** $\vec{r}=(\hat{i}-\hat{j}+2\hat{k})+\lambda(2\hat{i}+3\hat{j}-\hat{k})$]

18. Given line is $5x-25=14-7y=35z$.

$$\Rightarrow \frac{x-5}{1/5}=\frac{2-y}{1/7}=\frac{z}{1/35} \Rightarrow \frac{x-5}{1/5}=\frac{y-2}{-1/7}=\frac{z}{1/35}$$

⇒ Direction ratios of the given line are $\frac{1}{5}, -\frac{1}{7}, \frac{1}{35}$.

⇒ Direction ratios of a line parallel to the given line are proportional to $\frac{1}{5}, -\frac{1}{7}, \frac{1}{35}$.

∴ The required line will be parallel to the vector $\vec{b}=\frac{1}{5}\hat{i}-\frac{1}{7}\hat{j}+\frac{1}{35}\hat{k}$.

Hence, the required equation of line is given by $\vec{r}=\vec{a}+\lambda\vec{b}$

$$\Rightarrow \vec{r}=(\hat{i}+2\hat{j}-\hat{k})+\lambda\left(\frac{1}{5}\hat{i}-\frac{1}{7}\hat{j}+\frac{1}{35}\hat{k}\right)$$

19. Given equation of lines are

$$r=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-3\hat{k}) \quad \text{...(i)}$$

and $$r=(\hat{i}-\hat{j}+2\hat{k})+\mu(2\hat{i}+4\hat{j}-5\hat{k}) \quad \text{...(ii)}$$

On comparing Eqs. (i) and (ii) with $r=a_1+\lambda b_1$ and $r=a_2+\mu b_2$ respectively, we get

$$a_1=4\hat{i}-\hat{j},\ b_1=\hat{i}+2\hat{j}-3\hat{k}$$

and $$a_2=\hat{i}-\hat{j}+2\hat{k},\ b_2=2\hat{i}+4\hat{j}-5\hat{k}$$

Here $$a_2-a_1=-3\hat{i}+2\hat{k}$$

and $$b_1\times b_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix}$$

$$=\hat{i}(-10+12)-\hat{j}(-5+6)+\hat{k}(4-4)$$

$$=2\hat{i}-\hat{j}$$

$$\Rightarrow |b_1\times b_2|=\sqrt{2^2+(-1)^2}=\sqrt{4+1}=\sqrt{5}$$

Now, the shortest distance between the given lines is given by

$$d=\frac{|(b_1\times b_2)\cdot(a_2-a_1)|}{|b_1\times b_2|}=\frac{|(2\hat{i}-\hat{j})\cdot(-3\hat{i}+2\hat{k})|}{\sqrt{5}}$$

$$=\frac{|-6|}{\sqrt{5}}=\frac{6}{\sqrt{5}} \text{ units}$$

20. Given equation of line is

$$6x-2=3y+1=2z-2 \text{ or } \frac{6x-2}{6}=\frac{3y+1}{6}=\frac{2z-2}{6}$$

or $$\frac{6\left(x-\frac{2}{6}\right)}{6}=\frac{3\left(y+\frac{1}{3}\right)}{6}=\frac{2\left(z-\frac{2}{2}\right)}{6}$$

or $$\frac{x-\frac{1}{3}}{1}=\frac{y+\frac{1}{3}}{2}=\frac{z-1}{3}$$

Clearly, DR's of given line are 1, 2, 3.

$\therefore$ DC's of given line are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

$$[\because \sqrt{1^2+2^2+3^2}=\sqrt{1+4+9}=\sqrt{14}]$$

Now, equation of a line passing through the point $(2,-1,-1)$ and parallel to the given line is

$$\frac{x-2}{1}=\frac{y+1}{2}=\frac{z+1}{3}$$

To find the vector form, consider

$$\frac{x-2}{1}=\frac{y+1}{2}=\frac{z+1}{3}=\lambda \text{ (say)}$$

$\Rightarrow$ $x=\lambda+2,\ y=2\lambda-1,\ z=3\lambda-1$

$\therefore$ $x\hat{i}+y\hat{j}+z\hat{k}=(\lambda+2)\hat{i}+(2\lambda-1)\hat{j}+(3\lambda-1)\hat{k}$

$=(2\hat{i}-\hat{j}-\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$

$\Rightarrow$ $\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$

which is the required equation of line in vector form.

21. Solve as Question 20.

$\left[\text{Ans. } \left(\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}\right); \vec{r}=-\hat{i}+2\hat{j}+3\hat{k}+\lambda(2\hat{i}+3\hat{j}-6\hat{k})\right]$

22. Similar as Example 2.

[**Ans.** $\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$]

23. Similar as Example 2.

$\left[\text{Ans. } \vec{r}=(2\hat{i}-\hat{j}+3\hat{k})+\lambda(-6\hat{i}-3\hat{j}+6\hat{k}) \text{ and } \frac{2-x}{6}=\frac{-y-1}{3}=\frac{z-3}{6}\right]$

24. Similar as Example 5 (i). $\left[\text{Ans. } \theta=\cos^{-1}\left(\frac{19}{21}\right)\right]$

25. Similar as Example 5 (i). $\left[\text{Ans. } \frac{\pi}{3}\right]$

26. Similar as Example 5 (ii). $\left[\text{Ans. } \frac{\pi}{2}\text{; given pair of lines are perpendicular to each other}\right]$

27. **Hint** (i) The given equations of lines can be rewritten as

$$\frac{x-0}{3}=\frac{y-0}{2}=\frac{z-0}{-6} \text{ and } \frac{x-0}{2}=\frac{y-0}{-12}=\frac{z-0}{-3}.$$

(ii) Similar as Example 5 (ii). $\left[\text{Ans. } \theta=\frac{\pi}{2}\right]$

28. **Hint** Given equations of lines can be written as

$$\frac{x-1}{-3}=\frac{y-2}{2p/7}=\frac{z-3}{1} \quad \text{...(i)}$$

and $$\frac{x-1}{-3p/7}=\frac{y-5}{-1}=\frac{z-11}{-7} \quad \text{...(ii)}$$

Since, Eqs. (i) and (ii) are perpendicular.

$\therefore -3\left(\frac{-3p}{7}\right)+\frac{2p}{7}(-1)+1(-7)=0$ [**Ans.** $p=7$]

29. **Hint** If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of the lines, then the lines are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$.

30. Any line through the point (1, 1, 1) is given by

$$\frac{x-1}{a}=\frac{y-1}{b}=\frac{z-1}{c} \quad \text{...(i)}$$

where a, b and c are the direction ratios of line (i).

Now, the line (i) is perpendicular to the lines

$$\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$$

and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$, where DR's of these two lines are (1, 2, 4) and (2, 3, 4), respectively.

$\therefore$ $a+2b+4c=0$ ...(ii)

and $2a+3b+4c=0$ ...(iii)

[$\because$ if two lines having DR's (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular, then $a_1a_2+b_1b_2+c_1c_2=0$]

By cross-multiplication method, we get

$$\frac{a}{8-12}=\frac{b}{8-4}=\frac{c}{3-4} \Rightarrow \frac{a}{-4}=\frac{b}{4}=\frac{c}{-1}$$

$\therefore$ DR'S of line (i) are $-4, 4, -1$

$\therefore$ The required cartesian equation of line (i) is

$$\frac{x-1}{-4}=\frac{y-1}{4}=\frac{z-1}{-1}$$

and vector equation is $\vec{r}=\hat{i}+\hat{j}+\hat{k}+\lambda(-4\hat{i}+4\hat{j}-\hat{k})$

Again, let θ be the angle between the given lines. Then,

$$\cos\theta=\frac{1\times2+2\times3+4\times4}{\sqrt{1+4+16}\sqrt{4+9+16}}=\frac{24}{\sqrt{21}\sqrt{29}}=\frac{24}{\sqrt{609}}$$

$\therefore$ $\theta=\cos^{-1}\left(\frac{24}{\sqrt{609}}\right)$

31. Solve as Question 30.

Hint Find DR's of perpendicular lines.

$\left[\text{Ans. } \frac{x-1}{10}=\frac{y+1}{-4}=\frac{z-1}{-7}\text{; its corresponding vector equation is } \vec{r}=(\hat{i}-\hat{j}+\hat{k})+\lambda(10\hat{i}-4\hat{j}-7\hat{k})\right]$

32. Similar as Example 12. $\left[\text{Ans. } \frac{17}{5}\sqrt{5} \text{ units}\right]$

33. Similar as Example 13. [**Ans.** $2\sqrt{29}$ units]

34. Given equations of lines are

$$\vec{r} = (4\hat{i} + 5\hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$$

and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$

On comparing with $\vec{r} = a_1 + \lambda b_1$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$, we get $\vec{a_1} = 4\hat{i} + 5\hat{j}, \vec{b_1} = \hat{i} + 2\hat{j} - 3\hat{k}$

and $\vec{a_2} = \hat{i} - \hat{j} + 2\hat{k}, \vec{b_2} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Now, $\vec{a_2} - \vec{a_1} = \hat{i} - \hat{j} + 2\hat{k} - 4\hat{i} - 5\hat{j} = -3\hat{i} - 6\hat{j} + 2\hat{k}$

and $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$

$$= \hat{i}(-10 + 12) - \hat{j}(-5 + 6) + \hat{k}(4 - 4)$$

$$= 2\hat{i} - \hat{j} + 0\hat{k}$$

$\therefore \ |\vec{b_1} \times \vec{b_2}| = \sqrt{(2)^2 + (-1)^2 + (0)^2} = \sqrt{4 + 1} = \sqrt{5}$

$\because$ Shortest distance between two lines is

$$\text{SD} = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$\therefore \ \text{SD} = \left| \frac{(2\hat{i} - \hat{j} + 0\hat{k}) \cdot (-3\hat{i} - 6\hat{j} + 2\hat{k})}{\sqrt{5}} \right| = \left| \frac{-6 + 6 + 0}{\sqrt{5}} \right| = 0$$

Hence, the two lines intersect each other.

35. Given equations of lines are

$\dfrac{x-1}{3} = \dfrac{y+1}{2} = \dfrac{z-1}{5}$ and $\dfrac{x+2}{4} = \dfrac{y-1}{3} = \dfrac{z+1}{-2}$

On comparing above equations with

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$, we get

$$x_1 = 1, \ y_1 = -1, \ z_1 = 1, a_1 = 3, \ b_1 = 2, \ c_1 = 5$$

and $x_2 = -2, \ y_2 = 1, z_2 = -1, a_2 = 4, \ b_2 = 3, \ c_2 = -2$

Now, consider $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -3 & 2 & -2 \\ 3 & 2 & 5 \\ 4 & 3 & -2 \end{vmatrix}$

$$= -3(-4 - 15) - 2(-6 - 20) - 2(9 - 8)$$

$$= -3(-19) - 2(-26) - 2(1)$$

$$= 57 + 52 - 2 = 107 \neq 0$$

$\Rightarrow$ Shortest distance will not be zero.

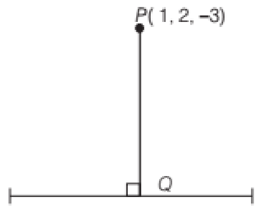
Hence, the given lines do not intersect each other.

36. Similar as Example 15 [**Ans.** (2, 6, – 2)]

37. Any point on the given line is $(2\lambda - 1, -2\lambda + 3, -\lambda)$.

Therefore, coordinates of Q are $(2\lambda - 1, -2\lambda + 3, -\lambda)$.

Now, $\overrightarrow{PQ} = (2\lambda - 2)\hat{i} + (-2\lambda + 1)\hat{j} + (-\lambda + 3)\hat{k}$



Since, $\overrightarrow{PQ}$ is perpendicular to the line

$$\frac{x+1}{2} = \frac{y-3}{-2} = \frac{z}{-1}$$

$\therefore \ 2(2\lambda - 2) - 2(-2\lambda + 1) - 1(-\lambda + 3) = 0$

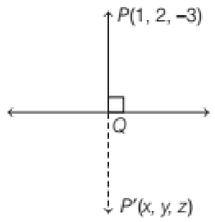
$\Rightarrow \qquad \lambda = 1$

$\therefore$ Foot of the perpendicular is $Q(1, 1, -1)$.

Let $P'(x, y, z)$ be the image of P in the line, then

Coordinates of Q = Coordinates of mid-point of PP'

$$\Rightarrow (1, 1, -1) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-3}{2} \right)$$



$\Rightarrow \qquad \dfrac{x+1}{2} = 1, \ \dfrac{y+2}{2} = 1$ and $\dfrac{z-3}{2} = -1$

$\Rightarrow x = 1, y = 0, z = 1$

Hence, the image is (1, 0, 1).

38. Solve as Question 37.

[**Ans.** Foot of perpendicular $= (-2, 1, 7)$ and image of $A = (-3, -6, 10)$.

39. Solve as Question 30.

[**Ans.** $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z+4}{6}$ and

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$]

40. The direction ratio of the lines

$\dfrac{x-1}{-3} = \dfrac{y-2}{2\lambda} = \dfrac{z-3}{2}$ and $\dfrac{x-1}{3\lambda} = \dfrac{y-1}{2} = \dfrac{z-6}{-5}$ are $-3, 2\lambda, 2$ and $3\lambda, 2, -5$, respectively.

It is known that two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular, if

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$\therefore \ (-3)(3\lambda) + (2\lambda)\,2 + (2)(-5) = 0$

$\Rightarrow \qquad -9\lambda + 4\lambda - 10 = 0$

$\Rightarrow \qquad -5\lambda = 10 \Rightarrow \lambda = -2$

Therefore, for $\lambda = -2$ the given lines are perpendicular.

The coordinates of any point on first line are given by

$$\frac{x-1}{-3}=\frac{y-2}{-4}=\frac{z-3}{2}=s \text{ (say)} \qquad [\because \lambda=-2]$$

$\Rightarrow \quad x=-3s+1,\ y=-4s+2$

and $\quad z=2s+3$

So, the coordinates of a general point on first line are

$$(-3s+1,\ -4s+2,\ 2s+3)$$

The coordinates of any point on second line are given by

$$\frac{x-1}{-6}=\frac{y-1}{2}=\frac{z-6}{-5}=t \text{ (say)}$$

$\Rightarrow \quad x=-6t+1,\ y=2t+1$ and $z=-5t+6$

So, the coordinates of a general point on second line are $(-6t+1,\ 2t+1,\ -5t+6)$

If the lines intersect, then they have a common point.
So, for some values of s and t, we must have,

$-3s+1=-6t+1,\ -4s+2=2t+1$

and $2s+3=-5t+6$

$\Rightarrow -3s+6t=0,\ -4s-2t=-1$ and $2s+5t=3$

Solving first two of these two equations, we get

$$s=\frac{1}{5} \text{ and } t=\frac{1}{10}$$

These values of s and t, do not satisfy the third equation.
Hence, the given lines do not intersect.

41. Similar as Example 9.

Hint The given equations can be rewritten as

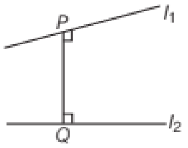
$$\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z+1}{0} \text{ and } \frac{x-4}{2}=\frac{y-0}{0}=\frac{z+1}{3}$$

[**Ans.** (4, 0, −1)]

42. Given lines are

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}=\lambda \text{ (say)} \qquad \text{...(i)}$$

and $$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}=\mu \text{ (say)} \qquad \text{...(ii)}$$



Any point P on line (i) is

$$P(3\lambda+8,\ -16\lambda-9,\ 7\lambda+10) \qquad \text{...(iii)}$$

and any point Q on line (ii) is

$$Q(3\mu+15,\ 8\mu+29,\ -5\mu+5) \qquad \text{...(iv)}$$

So, direction ratios of PQ are

$(3\mu+15-3\lambda-8,\ \ 8\mu+29+16\lambda+9,\ -5\mu+5-7\lambda-10)$

i.e. $(3\mu-3\lambda+7,\ 8\mu+16\lambda+38,\ -5\mu-7\lambda-5)$

Now, $|PQ|$ will be the shortest distance between lines (i) and (ii) if PQ is perpendicular to both lines (i) and (ii).

$\therefore \quad 3(3\mu-3\lambda+7)-16(8\mu+16\lambda+38)$
$+7(-5\mu-7\lambda-5)=0$
$[\because a_1a_2+b_1b_2+c_1c_2=0]$

$\Rightarrow 9\mu-9\lambda+21-128\mu-256\lambda-608-35\mu-49\lambda-35=0$

$\Rightarrow \quad -154\mu-314\lambda-622=0$

$\Rightarrow \quad 77\mu+157\lambda+311=0 \qquad$ [dividing by (-2)] ...(v)

and $3(3\mu-3\lambda+7)+8(8\mu+16\lambda+38)$
$-5(-5\mu-7\lambda-5)=0$
$[\because a_1a_2+b_1b_2+c_1c_2=0]$

$\Rightarrow 9\mu-9\lambda+21+64\mu+128\lambda+304+25\mu+35\lambda+25=0$

$\Rightarrow \quad 98\mu+154\lambda+350=0$

$\Rightarrow \quad 7\mu+11\lambda+25=0 \qquad$ [dividing by 14] ...(vi)

On multiplying Eq. (vi) by 11 and then subtracting from Eq. (v), we get

$(77\mu+157\lambda+311)-(77\mu+121\lambda+275)=0$

$\Rightarrow \quad 36\lambda+36=0 \Rightarrow \lambda=-1$

On putting the value of λ in Eq. (v), we get

$77\mu+157(-1)+311=0$

$\Rightarrow \quad 77\mu-157+311=0$

$\Rightarrow \quad 77\mu+154=0 \Rightarrow \mu=-2$

On putting the values of λ and μ in Eqs. (iii) and (iv), we get

Coordinates of $P=(-3+8,\ 16-9,\ -7+10)=(5,7,3)$

and coordinates of $Q=(-6+15,\ -16+29,\ 10+5)$
$=(9,13,15)$

$\therefore$ Shortest distance between two lines,

$$PQ=\sqrt{(9-5)^2+(13-7)^2+(15-3)^2}$$

$$[\because \text{distance}=(\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2})]$$

$$=\sqrt{16+36+144}=\sqrt{196}=14 \text{ units}$$

Equation of line PQ of shortest distance is

$$\frac{x-5}{9-5}=\frac{y-7}{13-7}=\frac{z-3}{15-3}$$

$$\left[\because \frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}\right]$$

$$\therefore \quad \frac{x-5}{4}=\frac{y-7}{6}=\frac{z-3}{12}$$

Hence, $\frac{x-5}{2}=\frac{y-7}{3}=\frac{z-3}{6}$ is the required equation of the line which gives shortest distance.

43. Similar as Example 15.

[**Ans.** Length of perpendicular is $\sqrt{24}$, coordinate of foot of perpendicular is $(3,-4,-2)$ and equation of perpendicular is $\frac{x-1}{1}=\frac{y}{-2}=\frac{z}{-1}$]

SUMMARY

- **Direction Cosines** A directed line say L passing through the origin makes angles α, β and γ with X, Y and Z-axes respectively, which are called direction angles. Then, cosine of these angles, i.e. $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as the direction cosines of the directed line L. We denote it as, $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.
- **Direction Ratios** Any three numbers a, b and c proportional to the direction cosines l, m and n, respectively are called the **direction ratios** or **direction numbers** of the line.
  (i) The direction ratios of a line passing through two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.
  (ii) $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$ and $n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$
- **Equation of a Line**
  (i) **Equation of a line through a given point and parallel to a given vector**
  Vector Equation $\vec{r} = \vec{a} + \lambda\vec{b}$, where, $\vec{a}$ = position vector of point and $\vec{b}$ vector to which line is parallel.
  Cartesian Equation $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$, where a, b, c are direction ratios and (x_1, y_1, z_1) be the point.
  (ii) **Equation of a line passing through two given points**
  Vector Equation $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, where $\vec{a}$ and $\vec{b}$ are position vectors of points.
  Cartesian Equation $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$, where (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points.
- **Angle between Two Lines**
  Vector form $\cos\theta = \left|\frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$.
  Cartesian form $\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$.
- **Distance between Two Skew-Lines** The shortest distance between these two skew-lines.
  Vector Form SD $= \left|\frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right|$
  Cartesian Form SD $= \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$
- **Distance between Parallel Lines** The shortest distance between parallel lines
  $$L_1 : \vec{r} = \vec{a_1} + \lambda\vec{b} \text{ and } L_2 : \vec{r} = \vec{a_2} + \mu\vec{b} \text{ is SD or } d = \left|\frac{\vec{b}\times(\vec{a_2}-\vec{a_1})}{|\vec{b}|}\right|$$

CHAPTER PRACTICE

OBJECTIVE TYPE QUESTIONS

1 P is a point on the line segment joining the points $(3, 2, -1)$ and $(6, 2, -2)$. If x-coordinate of P is 5, then its y-coordinate is **[NCERT Exemplar]**

(a) 2 (b) 1
(c) -1 (d) -2

2 If O is the origin and $OP = 3$ with direction ratios -1, 2 and -2, then coordinates of P are

(a) $(-1, 2, -2)$ (b) $\left(-\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$
(c) $(-3, 6, 9)$ (d) $(1, 2, 2)$

3 The equation of X-axis in space is **[NCERT Exemplar]**

(a) $x = 0, y = 0$ (b) $x = 0, z = 0$
(c) $x = 0$ (d) $y = 0, z = 0$

4 The coordinates of a point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ at a distance of $\frac{6}{\sqrt{2}}$ from the point $(1, 2, 3)$ is

(a) $(56, 43, 111)$ (b) $\left(\frac{56}{17}, \frac{43}{17}, \frac{111}{17}\right)$
(c) $(2, 1, 3)$ (d) $(-2, -1, -3)$

VERY SHORT ANSWER Type Questions

5 If a line makes angles 90°, 135°, 45° with the X, Y and Z-axes respectively, find its direction cosines. **[Delhi 2019]**

6 Find the direction cosines of the following line $\frac{3-x}{-1} = \frac{2y-1}{2} = \frac{z}{4}$ **[CBSE Sample Paper 2021 Term I]**

7 The line of shortest distance between two skew lines is .... to both the lines. **[All India 2020]**

8 Cartesian equation of line AB is $\frac{2x-1}{2} = \frac{4-y}{7} = \frac{z+1}{2}$. Write the direction ratios of a line parallel to AB. **[All India 2010]**

9 What is the cartesian equation of the line $\vec{r} = (3\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$? **[NCERT Exemplar]**

10 Write the vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}$. **[Delhi 2010]**

11 Find the equation of line passing through the point $(2, 1, 3)$ having the direction ratios $1, 1, -2$.

12 Find the equation of a line parallel to Y-axis and passing through the origin.

13 Write the condition for the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$ to be intersecting.

14 Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

SHORT ANSWER Type I Questions

15 If P is a point in space such that $OP = 12$ and OP is inclined at angles of 45° and 60° with X and Y-axes, respectively. Then, find the position vector of P.

[**Hint** First, use $l^2 + m^2 + n^2 = 1$]

16 Show that the line joining the origin to the point (2,1,1) is perpendicular to the line determined by the points $(3, 5, -1)$ and $(4, 3, -1)$. **[NCERT Exemplar]**

SHORT ANSWER Type II Questions

17 A line passing through the point A with position vector $\vec{a} = 4\hat{i} + 2\hat{j} + 2\hat{k}$ is parallel to the vector $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Find the length of the perpendicular drawn on this line from a point P with position vector $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k}$.

18 Find the vector and cartesian equations of line passing through the point (1, 2, – 4) and perpendicular to two lines

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}.$$

[Delhi 2012]

19 Show that the angle between the diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

20 Find the shortest distance between the lines

$$\vec{r}=(1-t)\,\hat{i}+(t-2)\,\hat{j}+(3-2t)\,\hat{k}$$

and $\vec{r}=(s+1)\,\hat{i}+(2s-1)\,\hat{j}-(2s+1)\,\hat{k}$.

[All India 2011]

21 Find the shortest distance between the lines $x+1=2y=-12z$ and $x=y+2=6z-6$.

22 Find the shortest distance between the following lines

$$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+s(2\hat{i}+\hat{j}+\hat{k})$$

and $\vec{r}=(\hat{i}+\hat{j}+2\hat{k})+t(4\hat{i}+2\hat{j}+2\hat{k})$

[CBSE Sample Paper (Term II)]

23 Show that the lines $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ and $\vec{r}=5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$ are intersecting. Also, find their point of intersecting.

[All India 2013]

LONG ANSWER Type Questions

24 If a variable line in two adjacent positions has direction cosines l, m, n and $l+\delta l, m+\delta m, n+\delta n$, then show that the small angle $\delta\theta$ between the two positions is given by $\delta\theta^2=\delta l^2+\delta m^2+\delta n^2$.

[NCERT Exemplar]

25 $\overrightarrow{AB}=3\hat{i}-\hat{j}+\hat{k}$ and $\overrightarrow{CD}=-3\hat{i}+2\hat{j}+4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i}+7\hat{j}+4\hat{k}$ and $-9\hat{j}+2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

[NCERT Exemplar]

26 Find the value of λ, so that the lines $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not.

[Delhi 2019]

27 A line makes angles α, β, γ *and* δ with the four diagonals of a cube, prove that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}.$$

CASE BASED Question

28 The equation of motion of a missile are $x=3t$, $y=-4t$ and $z=t$, where the time t is given in seconds and the distance is measured in kilometres.

[CBSE Question Bank]

Answer the following questions using the above information.

(i) What is the path of the missile?

(a) Straight line (b) Parabola
(c) Circle (d) Ellipse

(ii) Which of the following points lie on the path of the missile?

(a) (6, 8, 2) (b) (6, – 8, – 2)
(c) (6, – 8, 2) (d) (– 6, – 8, 2)

(iii) At what distance will the rocket be from the starting point (0, 0, 0) in 5 s?

(a) $\sqrt{550}$ km (b) $\sqrt{650}$ km
(c) $\sqrt{450}$ km (d) $\sqrt{750}$ km

ANSWERS

1. (a)
2. (a)
3. (d)
4. (b)
5. $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$
6. $\left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}\right)$
7. perpendicular
8. $(1, -7, 2)$
9. $\frac{x-3}{1} = \frac{y+1}{2} = \frac{z-4}{3}$
10. $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$
11. $\frac{x-2}{1} = \frac{y-1}{1} = \frac{z-3}{-2}$
12. $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$
13. $(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1}) = 0$
15. $6\sqrt{2}\hat{i} + 6\hat{j} + 6\hat{k}$
17. $\sqrt{10}$ units
18. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$; $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$
20. $\frac{8}{\sqrt{29}}$ units
21. 2 units
22. $\frac{3\sqrt{5}}{\sqrt{6}}$ units
23. $(-1, -6, -12)$
25. $P = (3, 8, 3)$, $Q = (-3, -7, 6)$; position vectors are $3\hat{i} + 8\hat{j} + 3\hat{k}$ and $-3\hat{i} - 7\hat{j} + 6\hat{k}$.
26. $\lambda = 7$, lines are not intersecting
28. (i) $\rightarrow$ (a), (ii) $\rightarrow$ (c), (iii) $\rightarrow$ (b)